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ON THE POSITIVE SOLUTIONS OF SYSTEMS OF DIFFERENCE EQUATIONS

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Abstract: We show that every positive solutions of the following system of difference equations

$$x_{n+1}^{(1)} = \frac{1}{x_n^{(2)}},\ x_{n+1}^{(2)} = \frac{1}{x_n^{(3)}},\ldots,\ x_{n+1}^{(k)} = \frac{1}{x_n^{(1)}}$$

as well as of the system

$$x_{n+1}^{(1)} = \frac{1}{x_n^{(k)}},\ x_{n+1}^{(2)} = \frac{1}{x_n^{(1)}},\ldots,\ x_{n+1}^{(k)} = \frac{1}{x_n^{(k-1)}}$$

is periodic with period equal to $2k$ if $k \neq 0(\mathrm{mod}2)$, and with period k if $k = 0\,(\mathrm{mod}\,2)$.

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1. Introduction

It is well known that all well defined solutions of the difference equation

$$x_{n+1} = \frac{1}{x_n} \tag{1.1}$$

are periodic with period two. Motivated by equation (1.1) we investigate

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the periodic character of the following two systems of difference equations, which can be considered as a natural generalizations of equation (1.1),

$$x_{n+1}^{(1)} = \frac{1}{x_n^{(2)}},\ x_{n+1}^{(2)} = \frac{1}{x_n^{(3)}}, ..., x_{n+1}^{(k)} = \frac{1}{x_n^{(1)}} \tag{1.2}$$

and

$$x_{n+1}^{(1)} = \frac{1}{x_n^{(k)}},\ x_{n+1}^{(2)} = \frac{1}{x_n^{(1)}}, ..., x_{n+1}^{(k)} = \frac{1}{x_n^{(k-1)}}. \tag{1.3}$$

In order to prove main results of the paper we need an auxiliary result which is contained in the following simple lemma from number theory. Let $GCD(k,l)$ denote the greatest common divisor of the integers k and l.

Lemma 1. *Let $k \in \mathbb{N}$, and $GCD(k,2) = 1$, then the numbers $a_l = 2l + 1$, (or $a_l = -2l + 1$), $l = 0, 1, ..., k - 1$, satisfy the following property $a_{l_1} - a_{l_2} \neq 0 \pmod k$, when $l_1 \neq l_2$.*

Proof. Suppose the contrary, then we have

$$2(l_1 - l_2) = a_{l_1} - a_{l_2} = ks\,,$$

for some $s \in \mathbb{Z}\setminus\{0\}$.

Since $GCD(k,2) = 1$, it follows that k is a divisor of $l_1 - l_2$. On the other hand, since $l_1, l_2 \in \{0, 1, ..., k-1\}$, we have $|l_1 - l_2| < k$, a contradiction. □

Remark 1. From Lemma 1 we see that the rests b_l, $l = 0, 1, ..., k-1$ of the numbers $a_l = 2l+1$, $l = 0, 1, ..., k-1$, obtained by dividing the numbers a_l by k are mutually different, they are contained in the set A=$\{0, 1, ..., k-1\}$, make a permutation of the ordered set $(0, 1, ..., k-1)$, and finally $a_k = 2k+1$ is the first number of the form $2l+1$, $l \in \mathbb{N}$, such that $a_1 - a_0 \equiv 0 \pmod k$.

2. Proofs of Main Results

In this section we formulate and prove the main results in this paper.

Theorem 1. *Consider equation (1.2), where $k \geq 1$. Then the following statements are true.*

(a) If $k \neq 0 \pmod 2$, then every positive solution of equation (1.2) is periodic with period $2k$.

(b) If $k = 0 \pmod 2$, then every positive solution of equation (1.2) is periodic with period k.

Proof. First note that the system is cyclic. Hence it is enough to prove that the sequence $(x_n^{(1)})$ satisfies conditions (a) and (b) in the corresponding cases.

Further, note that for every $s \in \mathbb{N}$ system (1.2) is equivalent to a system of ks difference equations of the same form, where

$$x_n^{(i)} = x_n^{(rk+i)}, \text{ for every } n \in \mathbb{N}, i \in \{1, ..., k\} \text{ and } r = 0, 1, ..., s-1. \quad (2.1)$$

On the other hand, we have

$$x_{n+1}^{(1)} = \frac{1}{x_n^{(2)}} = x_{n-1}^{(3)}. \quad (2.2)$$

(a) Let b_l, $l = 0, 1, ..., k-1$, are the rests mentioned in Remark 1. Then from (2.2) and Lemma 1 we obtain that

$$x_{n+1}^{(1)} = x_{n-1}^{(3)} = x_{n-3}^{(5)} = ... = x_{n+1-2(k-1)}^{(2k-1)} = x_{n+1-2k}^{(2k+1)}. \quad (2.3)$$

Using (2.1) for sufficently large s, we obtain that (2.3) is equivalent to (here we use the condition $GCD(k, 2) = 1$)

$$x_{n+1}^{(1)} = x_{n-1}^{(b_1)} = x_{n-3}^{(b_2)} = ... = x_{n+1-2(k-1)}^{(b_{k-1})} = x_{n+1-2k}^{(1)}.$$

From this and since by Lemma 1 the numbers $1, b_1, b_2, ..., b_{k-1}$ are piecewise different, the result follows in this case.

(b) Let $k = 2s$, for some $s \in \mathbb{N}$. By (2.1) we have

$$x_{n+1}^{(1)} = x_{n-1}^{(3)} = x_{n-3}^{(5)} = ... = x_{n+1-2s}^{(2s+1)} = x_{n+1-2s}^{(1)}$$

which yields the result. □

Remark 2. In order to make the proof of Theorem 1 clear to the reader, we explain what happens in the cases $k = 2$ and $k = 3$.

For $k = 2$ system (1.2) is equivalent to the system

$$\begin{aligned} &x_{n+1}^{(1)} = \frac{1}{x_n^{(2)}}, \quad x_{n+1}^{(2)} = \frac{1}{x_n^{(3)}}, \quad x_{n+1}^{(3)} = \frac{1}{x_n^{(4)}}, \quad x_{n+1}^{(4)} = \frac{1}{x_n^{(5)}}, \\ &x_{n+1}^{(5)} = \frac{1}{x_n^{(6)}}, \quad x_{n+1}^{(6)} = \frac{1}{x_n^{(1)}}, \end{aligned} \quad (2.4)$$

where we consider that $x_n^{(1)} = x_n^{(3)} = x_n^{(5)}$ and $x_n^{(2)} = x_n^{(4)} = x_n^{(6)}$, $n \in \mathbb{N}$.From this and (2.2) we have $x_{n+1}^{(1)} = x_{n-1}^{(3)} = x_{n-1}^{(1)}$ for every $n \in \mathbb{N}$. Using again

(2.2) we get $x_{n+1}^{(1)} = x_{n-1}^{(1)}$, which means that the sequence $x_n^{(1)}$ is periodic with period equal to 2.

If $k = 3$ system (1.2) is equivalent to system (2.4), where we consider that $x_n^{(1)} = x_n^{(4)}, x_n^{(2)} = x_n^{(5)}$ and $x_n^{(3)} = x_n^{(6)}$.Using this and (2.2) subsequently, it follows that

$$x_{n+1}^{(1)} = x_{n-1}^{(3)} = x_{n-3}^{(5)} = x_{n-3}^{(2)} = x_{n-5}^{(4)} = x_{n-5}^{(1)},$$

that is, the sequence $x_n^{(1)}$ is periodic with period 6.

Remark 3. The fact that every positive solution of equation (1.1) is periodic with period two can be considered as the case $k = 1$ in Theorem 1, that is, we can take that $x_n^{(1)} = x_n^{(2)} = ... = x_n^{(k)}$, for all $n \in \mathbb{N}$.

Similarly to Theorem 1, using Lemma 2 with $a_l = -2l+1, l = 0, 1, ..., k-1$, the following theorem can be proved.

Theorem 2. *Consider equation (1.3), where $k \geq 1$. Then the following statements are true.*

(a) If $k \neq 0(\text{mod}\, 2)$, then every positive solution of equation (1.3) is periodic with period $2k$.

(b) If $k = 0(\text{mod}\, 2)$, then every positive solution of equation (1.3) is periodic with period k.

Proof. First note that the system is cyclic. Hence it is enough to prove that the sequence $(x_n^{(1)})$ satisfies conditions (a) and (b) in the corresponding cases.

Indeed, similarly to (2.2), we have

$$x_{n+1}^{(1)} = \frac{1}{x_n^{(k)}} = x_{n-1}^{(k-1)}. \tag{2.5}$$

(a) Let b_l, $l = 0, 1, ..., k-1$, are the rests mentioned in Remark 1. Then from (2.5) and Lemma 1 we obtain that

$$x_{n+1}^{(1)} = x_{n-1}^{(k-1)} = x_{n-3}^{(k-3)} = ... = x_{n+1-2(k-1)}^{(-2k+3)} = x_{n+1-2k}^{(-2k+1)}. \tag{2.6}$$

Using (2.1) for sufficently large s, we obtain that (2.6) is equivalent to (here we use the condition $GCD(k, 2) = 1$)

$$x_{n+1}^{(1)} = x_{n-1}^{(b_1)} = x_{n-3}^{(b_2)} = ... = x_{n+1-2(k-1)}^{(b_{k-1})} = x_{n+1-2k}^{(1)}.$$

From this and since by Lemma 1 the numbers $1, b_1, b_2, ..., b_{k-1}$ are piecewise different, the result follows in this case.

(b) Let $k = 2s$, for some $s \in \mathbb{N}$. By (2.1) we have

$$x_{n+1}^{(1)} = x_{n-1}^{(k-1)} = x_{n-3}^{(k-3)} = ... = x_{n+1-2s}^{(1)}.$$

which yields the result. □

Example 1. Let $k = 3$. Then the solution of equation (1.2) with the initial conditions $x_0 = p$, $y_0 = q$ and $z_0 = r$, in its invertal of periodicity can be represented by the following table:

i	1	2	3	4	5	6	7	8	9	10	11	12
x_i	$\frac{1}{q}$	r	$\frac{1}{p}$	q	$\frac{1}{r}$	p	$\frac{1}{q}$	r	$\frac{1}{p}$	q	$\frac{1}{r}$	p
y_i	$\frac{1}{r}$	p	$\frac{1}{q}$	r	$\frac{1}{p}$	q	$\frac{1}{r}$	p	$\frac{1}{q}$	r	$\frac{1}{p}$	q
z_i	$\frac{1}{p}$	q	$\frac{1}{r}$	p	$\frac{1}{q}$	r	$\frac{1}{p}$	q	$\frac{1}{r}$	p	$\frac{1}{q}$	r

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