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ON THE ASYMPTOTICALLY PERIODIC SOLUTIONS OF A FIFTH ORDER DIFFERENCE EQUATION

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Abstract. In this paper we investigate the stability of solutions of difference equation $x_{n+1} = x_{n-1}x_{n-4} - 1$. We also study periodic solutions of related difference equation especially asymptotic periodicity.

1. Introduction

The existence of periodic solutions of difference equations of higher order has a significant place in the applied sciences. Moreover, the stability analysis of difference equations has been studied with great interest by many researchers, over the last years. One of the most important reasons for this interest is that difference equations have important applications for real life situations. Many mathematical models created with difference equations apply to various field of science such as population, genetic and economy. In addition, many authors have worked on the relevant articles on this topic. Though difference equations seem to simple appearance form, but it is quite difficult to understand thoroughly the behaviors of their solutions, see [1]-[34] and the references cited therein. Periodic solutions of nonlinear difference equations of higher order are studied their asymptotic properties, by many authors such as Berg in [21], Karakostas in [13], Stevic in [30], Grove and Ladas in [11].

In this study we deal with the stability of solutions of the equilibrium point of the following difference equation

$$x_{n+1} = x_{n-1}x_{n-4} - 1, \quad n = 0, 1, \cdots, \tag{1.1}$$

with arbitrary positive initial conditions x_{-4}, x_{-3}, x_{-2}, x_{-1}, x_0. Moreover, we investigate the periodic and asymptotically periodic solutions of related difference equation. Eq.(1.1) be a member of the class of equations of the form

$$x_{n+1} = x_{n-k}x_{n-l} - 1, \; n = 0, 1, \cdots, \tag{1.2}$$

with special choices of k and l, where $k, l \in \mathbb{N}_0$. Additionally, solutions of various forms of Eq.(1.2) had been studied by many authors in [2] - [5], [8] - [10], [20], [34].

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Now, we present some important definitions and theorems for difference equations.

Definition 1.1 (Difference Equation, [6]). *A **difference equation** of order* $(k+1)$ *is an equation of the form*

$$x_{n+1} = F(x_n, x_{n-1}, ..., x_{n-k}), \ n = 0, 1, \dots \tag{1.3}$$

where F *is a function that maps some set* I^{k+1} *into* I*. The set* I *is usually an interval of real numbers, or a union of intervals, or a discrete set such as the set of integers* $\mathbb{Z} = \{\dots, -1, 0, 1, \dots\}$.
A solution of Eq.(1.3) is a sequence $\{x_n\}_{n=-k}^{\infty}$ *that satisfies Eq.(1.3) for all* $n \geq 0$.

Definition 1.2 (Equilibrium, [6]). *A solution of Eq.(1.3) that is constant for all* $n \geq -k$ *is called an **equilibrium solution** of Eq.(1.3). If*

$$x_n = \overline{x}, \textit{for all } n \geq -k$$

is an equilibrium solution of Eq.(1.3), then $\overline{x}$ *is called an **equilibrium point**, or simply an **equilibrium** of Eq.(1.3).*

Definition 1.3 (Stability, [6]). *(i) An equilibrium point* $\overline{x}$ *of Eq.(1.3) is called **locally stable** if, every* $\varepsilon > 0$*, there exists* $\delta > 0$ *such that if* $\{x_n\}_{n=-k}^{\infty}$ *is a solution of Eq.(1.3) with*

$$|x_{-k} - \bar{x}| + |x_{1-k} - \bar{x}| + \cdots + |x_0 - \bar{x}| < \delta$$

then

$$|x_n - \bar{x}| < \varepsilon, \textit{ for all } n \geq 0.$$

(ii) An equilibrium point $\overline{x}$ *of Eq.(1.3) is called **locally asymptotically stable** if,* $\bar{x}$ *is locally stable, and if addition there exists* $\gamma > 0$ *such that if* $\{x_n\}_{n=-k}^{\infty}$ *is a solution of Eq.(1.3) with*

$$|x_{-k} - \bar{x}| + |x_{1-k} - \bar{x}| + \cdots + |x_0 - \bar{x}| < \gamma,$$

then

$$\lim_{n\to\infty} x_n = \bar{x}.$$

(iii) An equilibrium point $\overline{x}$ *of Eq.(1.3) is called a **global attractor** if, for every solution* $\{x_n\}_{n=-k}^{\infty}$ *of Eq.(1.3), we have*

$$\lim_{n\to\infty} x_n = \bar{x}.$$

(iv) An equilibrium point $\overline{x}$ *of Eq.(1.3) is called **globally asymptotically stable** if* $\bar{x}$ *is locally stable, and* $\bar{x}$ *is also a global attractor of Eq.(1.3).*
(v) An equilibrium point $\overline{x}$ *of Eq.(1.3) is called **unstable** if* $\bar{x}$ *is not locally stable.*

Definition 1.4 (Linearized Equation, [6]). *Suppose that the function* F *is continuously differentiable in some open neighborhood of an equilibrium point* $\bar{x}$*. Let*

$$q_i = \frac{\partial F}{\partial u_i}\left(\bar{x}, \bar{x}, \cdots, \bar{x}\right), \textit{ for } i = 0, 1, 2, \cdots, k$$

denote the partial deriative of $F\left(u_0, u_1, \cdots, u_k\right)$ *with respect to* u_i *evaluated at the equilibrium point* $\bar{x}$ *of Eq.(1.3). The equation*

$$z_{n+1} = q_0 z_n + q_1 z_{n-1} + \cdots + q_k z_{n-k}, \, n = 0, 1, \cdots, \tag{1.4}$$

*is called the **linearized equation** of Eq.(1.3) about the equilibrium point* $\bar{x}$.

Definition 1.5 (Characteristic Equation, [6]). *The equation*

$$\lambda^{k+1} - q_0\lambda^k - q_1\lambda^{k-1} - \cdots - q_{k-1}\lambda - q_k = 0 \tag{1.5}$$

*is called the **characteristic equation** of Eq.(1.4) about the equilibrium point $\bar{x}$.*

Theorem 1.6 (The Linearized Stability Theorem, [6]). *Assume that the function F is a continuously differentiable function defined on some open neighborhood of an equilibrium point $\bar{x}$. Then the following statements are true:*

*(a) When all the roots of Eq.(1.5) have absolute value less than one, then the equilibrium point $\bar{x}$ of Eq.(1.3) is **locally asymptotically stable.** Moreover, in this here the equilibrium point $\bar{x}$ of Eq.(1.3) is called **sink**.*

*(b) If at least one root of Eq.(1.5) has absolute value greater than one, then the equilibrium point $\bar{x}$ of Eq.(1.3) is **unstable.***

*(i) The equilibrium point $\bar{x}$ of Eq.(1.3) is called **hyperbolic** if no root of Eq.(1.5) has absolute value equal to one.*

*(ii) If there exists a root of Eq.(1.5) with absolute value equal to one, then the equilibrium $\bar{x}$ is called **nonhyperbolic.***

*(iii) An equilibrium point $\bar{x}$ of Eq.(1.3) is called a **saddle point** if it is hyperbolic and if there exists a root of Eq.(1.5) with absolute value less than one and another root of Eq.(1.5) with absolute value greater than one.*

*(iv) An equilibrium point $\bar{x}$ of Eq.(1.3) is called a **repeller** if all roots of Eq.(1.5) have absolute value greater than one.*

Definition 1.7 (Periodicity, [6]). *A solution $\{x_n\}_{n=-k}^{\infty}$ of Eq.(1.3) is called **periodic** with period p if there exists an integer $p \geq 1$ such that*

$$x_{n+p} = x_n, \text{ for all } n \geq -k. \tag{1.6}$$

*A solution is called periodic with **prime period** p if p is the smallest positive integer for which Eq.(1.6) holds.*

Definition 1.8 (Asymptotically Periodic, [19]). *A solution of $\{x_n\}_{n=-k}^{\infty}$ of Eq.(1.3) is called asymptotically periodic (asymptotically p-periodic) if there exist two sequences $u, w : \mathbb{N} \to \mathbb{R}$ such that u is periodic (p-periodic), $\lim_{n\to\infty} w_n = 0$ and $x_n = u_n + w_n$ for all $n \in \mathbb{N}$.*

2. Stability Analysis of Eq.(1.1)

In this section, firstly we investigate the equilibrium points of Eq.(1.1). Then, we study the stability analysis of Eq.(1.1).

Lemma 2.1. *The equilibrium points of Eq.(1.1) are*

$$\bar{x}_{1,2} = \frac{1 \pm \sqrt{5}}{2}. \tag{2.1}$$

Note that the equilibrium points are golden number ($\frac{1+\sqrt{5}}{2} = 1.618$) and its conjugate.

Proof. Let $x_n = \overline{x}$ for all $n \geq -4$. Therefore, we get from Eq.(1.1)

$$\begin{aligned} \bar{x} &= \bar{x} \cdot \bar{x} - 1 && (2.2)\\ \bar{x}^2 - \bar{x} - 1 &= 0. && (2.3) \end{aligned}$$

Thus, from (2.3),

$$\bar{x}_{1,2} = \frac{1 \pm \sqrt{5}}{2}.$$

So, we obtain the negative and positive equilibrium points of Eq.(1.1) together. □

Lemma 2.2. *The linearized equation of Eq.(1.1) about its equilibrium point* $\overline{x}$ *is*

$$z_{n+1} - \overline{x} \cdot z_{n-1} - \overline{x} \cdot z_{n-4} = 0. \tag{2.4}$$

Proof. Let I be some interval of real numbers and let

$$f : I^5 \to I$$

be a continuously differentiable function such that f is defined by

$$f\left(x_n, x_{n-1}, x_{n-2}, x_{n-3}, x_{n-4}\right) = x_{n-1}x_{n-4} - 1.$$

Thus we obtain,

$$q_0 = \frac{\partial f}{\partial x_n}\left(\overline{x}, \overline{x}, \overline{x}, \overline{x}, \overline{x}\right) = [0]\left(\overline{x}, \overline{x}, \overline{x}, \overline{x}, \overline{x}\right) = 0,$$

$$q_1 = \frac{\partial f}{\partial x_{n-1}}\left(\overline{x}, \overline{x}, \overline{x}, \overline{x}, \overline{x}\right) = [x_{n-4}]\left(\overline{x}, \overline{x}, \overline{x}, \overline{x}, \overline{x}\right) = \overline{x},$$

$$q_2 = \frac{\partial f}{\partial x_{n-2}}\left(\overline{x}, \overline{x}, \overline{x}, \overline{x}, \overline{x}\right) = [0]\left(\overline{x}, \overline{x}, \overline{x}, \overline{x}, \overline{x}\right) = 0,$$

$$q_3 = \frac{\partial f}{\partial x_{n-3}}\left(\overline{x}, \overline{x}, \overline{x}, \overline{x}, \overline{x}\right) = [0]\left(\overline{x}, \overline{x}, \overline{x}, \overline{x}, \overline{x}\right) = 0,$$

$$q_4 = \frac{\partial f}{\partial x_{n-4}}\left(\overline{x}, \overline{x}, \overline{x}, \overline{x}, \overline{x}\right) = [x_{n-1}]\left(\overline{x}, \overline{x}, \overline{x}, \overline{x}, \overline{x}\right) = \bar{x}.$$

If $\overline{x}$ denotes an equilibrium point of Eq.(1.1), then the linearized equation associated with Eq.(1.1) about the equilibrium point $\overline{x}$ is

$$z_{n+1} = q_0 \cdot z_n + q_1 \cdot z_{n-1} + q_2 \cdot z_{n-2} + q_3 \cdot z_{n-3} + q_4 \cdot z_{n-4}$$

then

$$z_{n+1} - \overline{x} \cdot z_{n-1} - \overline{x} \cdot z_{n-4} = 0.$$

So, the proof is completed. □

Lemma 2.3. *The characteristic equation of Eq.(1.1) about its equilibrium point* $\overline{x}$ *is*

$$\lambda^5 - \overline{x} \cdot \lambda^3 - \overline{x} = 0. \tag{2.5}$$

Proof. We have from linearized equation of Eq.(1.1) that

$$\lambda^5 - \overline{x} \cdot \lambda^3 - \overline{x} = 0.$$

□

Now, we investigate the stability of negative and positive equilibrium points of Eq.(1.1).

Theorem 2.4. *The negative equilibrium* $\bar{x}_1 = \frac{1-\sqrt{5}}{2}$ *of Eq.(1.1) is unstable. Its a saddle point.*

Proof. We consider (2.5) for $\bar{x}_1 = \frac{1-\sqrt{5}}{2}$. Therefore, we get following five roots of (2.5):

$$\begin{aligned}\lambda_1 &\approx -0.791\\ \lambda_{2,3} &\approx -0.223 \pm 1.006i\\ \lambda_{4,5} &\approx 0.619 \pm 0.594i.\end{aligned}$$

Hence,

$$\begin{aligned}|\lambda_1| &\approx 0.79 < 1\\ |\lambda_{2,3}| &\approx 1.03 > 1\\ |\lambda_{4,5}| &\approx 0.86 < 1.\end{aligned}$$

Thus, we obtain that

$$|\lambda_1| < |\lambda_{4,5}| < 1 < |\lambda_{2,3}|.$$

So, the negative equilibrium $\bar{x}_1 = \frac{1-\sqrt{5}}{2}$ of Eq.(1.1) is unstable. Because it is a saddle point. □

Theorem 2.5. *The positive equilibrium $\bar{x}_2 = \frac{1+\sqrt{5}}{2}$ of Eq.(1.1) is unstable. Its a saddle point.*

Proof. We consider (2.5) for $\bar{x}_2 = \frac{1+\sqrt{5}}{2}$. Thus, we obtain following five roots of (2.5):

$$\begin{aligned}\lambda_1 &\approx 1.462\\ \lambda_{2,3} &\approx -1.106 \pm 0.384i\\ \lambda_{4,5} &\approx 0.375 \pm 0.817i.\end{aligned}$$

Therefore,

$$\begin{aligned}|\lambda_1| &\approx 1.46 > 1\\ |\lambda_{2,3}| &\approx 1.17 > 1\\ |\lambda_{4,5}| &\approx 0.89 < 1.\end{aligned}$$

Thus, we obtain that

$$|\lambda_{4,5}| < 1 < |\lambda_{2,3}| < |\lambda_1|.$$

Hence, the positive equilibrium $\bar{x}_2 = \frac{1+\sqrt{5}}{2}$ of Eq.(1.1) is unstable. Because it is a saddle point. □

3. Existence of Periodic Solutions of Eq.(1.1)

Going throughout this section, we investigate the existence of periodic solutions of Eq.(1.1).

Theorem 3.1. *There are no periodic solutions of Eq.(1.1) with period two.*

Proof. Suppose that α, β are real numbers, $\alpha \neq \beta$ and

$$\cdots, \alpha, \beta, \alpha, \beta, \alpha, \cdots$$

is a periodic solution of Eq.(1.1) with period two. Then we get from Eq.(1.1) that

$$\alpha = \alpha \cdot \beta - 1, \tag{3.1}$$

$$\beta = \beta \cdot \alpha - 1. \tag{3.2}$$

From (3.1) and (3.2), we obtain $\alpha = \beta = \bar{x}_1$ or $\alpha = \beta = \bar{x}_2$. Due to its trivial solution of Eq.(1.1), Eq.(1.1) has not periodic solution with period two. $\square$

Note that proofs of Theorem 3.2-Theorem 3.5 are similar to the proof of Theorem 3.1, thus we leave them to the readers.

Theorem 3.2. *There are no periodic solutions of Eq.(1.1) with period three.*

Theorem 3.3. *There are no periodic solutions of Eq.(1.1) with period four.*

Theorem 3.4. *Eq.(1.1) has no periodic solutions with period five.*

Theorem 3.5. *Eq.(1.1) has no periodic solutions with period six.*

Theorem 3.6. *Eq.(1.1) has periodic solutions with prime period seven.*

Proof. Let $\alpha, \beta, \gamma, \phi, \varphi, \psi, \theta$ are real numbers such that at least two are different from each other. Assume that a solution $\{x_n\}_{n=-4}^{\infty}$ is of period seven. Therefore, we have from Eq.(1.1)

$$\begin{aligned}
x_{-4} &= \alpha, \\
x_{-3} &= \beta, \\
x_{-2} &= \gamma, \\
x_{-1} &= \phi, \\
x_{0} &= \varphi, \\
x_{1} &= \psi = \phi \cdot \alpha - 1, \\
x_{2} &= \theta = \varphi \cdot \beta - 1, \\
x_{3} &= \alpha = \psi \cdot \gamma - 1 = (\phi \cdot \alpha - 1)\,\gamma - 1 = \alpha\gamma\phi - \gamma - 1, && (3.3)\\
x_{4} &= \beta = \theta \cdot \phi - 1 = (\varphi \cdot \beta - 1)\,\phi - 1 = \beta\phi\varphi - \phi - 1, && (3.4)\\
x_{5} &= \gamma = \alpha \cdot \varphi - 1, && (3.5)\\
x_{6} &= \phi = \beta \cdot \psi - 1 = \beta\,(\phi \cdot \alpha - 1) - 1 = \alpha\beta\phi - \beta - 1, && (3.6)\\
x_{7} &= \varphi = \gamma \cdot \theta - 1 = \gamma\,(\varphi \cdot \beta - 1) - 1 = \beta\gamma\varphi - \gamma - 1. && (3.7)
\end{aligned}$$

Hence, we obtain from (3.5):

$$\begin{aligned}
x_{3} &= \alpha = \alpha\gamma\phi - \gamma - 1 = \alpha\,(\alpha\phi\varphi - \phi - \varphi)\,, && (3.8)\\
x_{4} &= \beta = \beta\phi\varphi - \phi - 1, && (3.9)\\
x_{6} &= \phi = \alpha\beta\phi - \beta - 1, && (3.10)\\
x_{7} &= \varphi = \beta\gamma\varphi - \gamma - 1 = \varphi\,(\alpha\beta\varphi - \alpha - \beta)\,. && (3.11)
\end{aligned}$$

Thus, we get from (3.8)-(3.11):

$$\begin{aligned}
\alpha\,(\alpha\phi\varphi - \phi - \varphi - 1) &= 0, && (3.12)\\
\varphi\,(\alpha\beta\varphi - \alpha - \beta - 1) &= 0, && (3.13)\\
\beta\phi\varphi - \phi - \beta - 1 &= 0, && (3.14)\\
\alpha\beta\phi - \beta - \phi - 1 &= 0. && (3.15)
\end{aligned}$$

So, we obtain from (3.14)-(3.15):

$$\begin{aligned}
\beta\phi\varphi - \alpha\beta\phi &= 0, && (3.16)\\
\beta\phi\,(\varphi - \alpha) &= 0. && (3.17)
\end{aligned}$$

From (3.17), we have three cases such that $\beta = 0$ or $\phi = 0$ or $\varphi = \alpha$.

Case 1. Let $\beta = 0$. From (3.14), we obtain $\phi = -1$. Hence, we get from (3.12):

$$\begin{aligned} \alpha\left(\alpha\phi\varphi - \phi - \varphi - 1\right) &= 0, \\ \alpha\left(-\alpha\varphi + 1 - \varphi - 1\right) &= 0, \\ \alpha\left(-\alpha\varphi - \varphi\right) &= 0, \\ \alpha\varphi\left(\alpha + 1\right) &= 0. \end{aligned}$$

Thus, we have the following three cases that $\alpha = 0$ or $\alpha = -1$ or $\varphi = 0$. So, we can constitute the following table from these resuls and (3.5):

	Seven periodic cycles of Eq.(1.1)
$\alpha = 0, \beta = 0, \gamma = -1, \phi = -1 \Rightarrow$	$(x_n)_{n=1}^{\infty} = (-1, -1, 0, 0, -1, -1, 0, \cdots)$
$\alpha = -1, \beta = 0, \phi = -1 \Rightarrow$	$(x_n)_{n=1}^{\infty} = (0, -1, -1, 0, -\varphi - 1, -1, \varphi, \cdots)$
$\beta = 0, \gamma = -1, \varphi = 0, \phi = -1 \Rightarrow$	$(x_n)_{n=1}^{\infty} = (-\alpha - 1, -1, \alpha, 0, -1, -1, 0, \cdots)$

Case 2. Let $\phi = 0$. From (3.14), we have $\beta = -1$. Therefore we obtain from (3.13):

$$\begin{aligned} \varphi\left(\alpha\beta\varphi - \alpha - \beta - 1\right) &= 0 \\ \varphi\left(-\alpha\varphi - \alpha + 1 - 1\right) &= 0, \\ \varphi\left(-\alpha\varphi - \alpha\right) &= 0, \\ \varphi\alpha\left(\varphi + 1\right) &= 0. \end{aligned}$$

So, we get the three cases $\varphi = 0$ or $\varphi = -1$ or $\alpha = 0$. Therefore, we can arrange the table from these results and (3.5):

	Seven periodic cycles of Eq.(1.1)
$\beta = -1, \gamma = -1, \phi = 0, \varphi = 0 \Rightarrow$	$(x_n)_{n=1}^{\infty} = (-1, -1, 0, -1, -1, 0, 0, \cdots)$
$\beta = -1, \phi = 0, \varphi = -1, \alpha + \gamma = -1 \Rightarrow$	$(x_n)_{n=1}^{\infty} = (-1, 0, \alpha, -1, -\alpha - 1, 0, -1, \cdots)$
$\alpha = 0, \beta = -1, \gamma = -1, \phi = 0 \Rightarrow$	$(x_n)_{n=1}^{\infty} = (-1, -\varphi - 1, 0, -1, -1, 0, \varphi, \cdots)$

Case 3. Let $\varphi = \alpha$. Hence we obtain followings from (3.12) and (3.13):

$$\begin{aligned} \alpha\left(\alpha\phi\varphi - \phi - \varphi - 1\right) &= 0 \\ &\Rightarrow \alpha\left(\alpha^2\phi - \phi - \alpha - 1\right) = 0 \\ &\Rightarrow \alpha\left(\alpha + 1\right)\left(\phi\left(\alpha - 1\right) - 1\right) = 0 \qquad (3.18) \\ \varphi\left(\alpha\beta\varphi - \alpha - \beta - 1\right) &= 0 \\ &\Rightarrow \alpha\left(\alpha^2\beta - \alpha - \beta - 1\right) = 0 \\ &\Rightarrow \alpha\left(\alpha + 1\right)\left(\beta\left(\alpha - 1\right) - 1\right) = 0 \qquad (3.19) \end{aligned}$$

From (3.18) and (3.19), we have $\alpha = 0$ or $\alpha = -1$ or $\phi = \beta = \frac{1}{\alpha - 1}$.

From (3.15), if $\alpha = \varphi = 0$ then $\beta + \phi = -1$. From (3.15), if $\alpha = \varphi = -1$ then $\beta = -1$ or $\phi = -1$. If $\phi = \beta = \frac{1}{\alpha - 1}$, we have from (3.15):

$$\begin{aligned} \alpha\beta\phi - \beta - \phi - 1 &= 0, \\ \alpha\left(\frac{1}{\alpha - 1}\right)^2 - \frac{1}{\alpha - 1} - \frac{1}{\alpha - 1} - 1 &= 0, \\ \frac{\alpha^2 - \alpha - 1}{\left(\alpha - 1\right)^2} &= 0. \end{aligned}$$

Hence, we obtain $\alpha = \frac{1-\sqrt{5}}{2}$ or $\alpha = \frac{1+\sqrt{5}}{2}$. Therefore, we get from $\phi = \beta = \frac{1}{\alpha-1}$ and (3.5):

$$\alpha = \beta = \gamma = \varphi = \phi = \frac{1-\sqrt{5}}{2}, \tag{3.20}$$

$$\alpha = \beta = \gamma = \varphi = \phi = \frac{1+\sqrt{5}}{2}. \tag{3.21}$$

Note that the last conditions (3.20) and (3.21) are equilibrium solutions. Thus, they are not periodic solutions with period seven.

Hence, we can create the table from these results and (3.5):

	Seven periodic cycles of Eq.(1.1)
$\alpha = \varphi = 0, \gamma = -1, \beta + \phi = -1 \Rightarrow$	$(x_n)_{n=1}^{\infty} = (-1, -1, 0, -\phi - 1, -1, \phi, 0, \cdots)$
$\alpha = \varphi = -1, \gamma = 0, \beta = -1 \Rightarrow$	$(x_n)_{n=1}^{\infty} = (-\phi - 1, 0, -1, -1, 0, \phi, -1, \cdots)$
$\alpha = \varphi = -1, \phi = -1, \gamma = 0 \Rightarrow$	$(x_n)_{n=1}^{\infty} = (0, -\beta - 1, -1, \beta, 0, -1, -1, \cdots)$

So, Eq.(1.1) has seven periodic solutions as desired. □

The following theorem has an important role for all type of Eq.(1.2) because Eq.(1.2) possesses an invariant interval.

Theorem 3.7 (See [2], Theorem 4.1)**.** *Let* $\{x_n\}_{n=-k}^{\infty}$ *be a solution of Eq.(1.2). If* $x_{-k}, x_{-k+1}, ..., x_{-1}, x_0 \in (-1, 0)$*, then* $x_n \in (-1, 0)$ *for all* $n \geq -k$.

4. Existence of Asymptotically Seven Periodic Solutions of Eq.(1.1)

In this section, we search for an asymptotically periodic solution with period seven. For an arbitrary m, Eq.(1.1) possesses the seven periodic solution

$$\begin{aligned} x_{7n-4} &= 0, \\ x_{7n-3} &= -1, \\ x_{7n-2} &= -1, \\ x_{7n-1} &= 0, \\ x_{7n} &= -m-1, \\ x_{7n+1} &= -1, \\ x_{7n+2} &= m, \end{aligned}$$

as Theorem 3.6.

Theorem 4.1. *If the initial conditions in* $(-1, 0)$ *and* $m \in (-1, 0)$*, then Eq.(1.1) has an asymptotically seven periodic solution.*

Proof. Firstly, we set up the approximation seven periodic cycle, up to an $O(t^{2n})$;

$$x_{7n-4} = 0 + a \cdot t^n, \tag{4.1}$$

$$x_{7n-3} = -1 + b \cdot t^n, \tag{4.2}$$

$$x_{7n-2} = -1 + c \cdot t^n, \tag{4.3}$$

$$x_{7n-1} = 0 + d \cdot t^n, \tag{4.4}$$

$$x_{7n} = -m - 1 + e \cdot t^n, \tag{4.5}$$

$$x_{7n+1} = -1 + f \cdot t^n, \tag{4.6}$$

$$x_{7n+2} = m + k \cdot t^n. \tag{4.7}$$

Hence, the coefficients of (4.1) - (4.7) must satisfy the following equations for Eq.(1.1)

$$\begin{aligned}
-1+f\cdot t^n &= (0+d\cdot t^n)(0+a\cdot t^n)-1, && (4.8)\\
m+k\cdot t^n &= (-m-1+e\cdot t^n)(-1+b\cdot t^n)-1, && (4.9)\\
0+a\cdot t^{n+1} &= (-1+f\cdot t^n)(-1+c\cdot t^n)-1, && (4.10)\\
-1+b\cdot t^{n+1} &= (m+k\cdot t^n)(0+d\cdot t^n)-1, && (4.11)\\
-1+c\cdot t^{n+1} &= (0+a\cdot t^{n+1})(-m-1+e\cdot t^n)-1, && (4.12)\\
0+d\cdot t^{n+1} &= (-1+b\cdot t^{n+1})(-1+f\cdot t^n)-1, && (4.13)\\
-m-1+e\cdot t^{n+1} &= (-1+c\cdot t^{n+1})(m+k\cdot t^n)-1. && (4.14)
\end{aligned}$$

From (4.8) - (4.14), again up to $O(t^{2n})$, we obtain that

$$\begin{aligned}
f &= 0, && (4.15)\\
(m+1)b+e+k &= 0, && (4.16)\\
at+c+f &= 0, && (4.17)\\
bt-md &= 0, && (4.18)\\
(m+1)ta+ct &= 0, && (4.19)\\
tb+dt+f &= 0, && (4.20)\\
-mtc+et+k &= 0. && (4.21)
\end{aligned}$$

The homogeneous system given by (4.15) - (4.21) can be solvable, if its determinant

$$\begin{vmatrix}
0 & 0 & 0 & 0 & 0 & 1 & 0\\
0 & m+1 & 0 & 0 & 1 & 0 & 1\\
t & 0 & 1 & 0 & 0 & 1 & 0\\
0 & t & 0 & -m & 0 & 0 & 0\\
(m+1)t & 0 & t & 0 & 0 & 0 & 0\\
0 & t & 0 & t & 0 & 1 & 0\\
0 & 0 & -mt & 0 & t & 0 & 1
\end{vmatrix}$$

is equal to zero. Because it must be $|t| < 1$, we hope the existence of an asymptotically seven periodic solution with the asymptotic approximations (4.1) - (4.7). If this determinant is calculated, then the result is

$$t(t^4-2t^3+(1-m-m^2)t^2+(m+m^2)t)=0. \tag{4.22}$$

Therefore, we have for $m \in (-1,0)$,

$$\begin{aligned}
t_1 &= 0, && (4.23)\\
t_2 &= 0, && (4.24)\\
t_3 &= -m, && (4.25)\\
t_4 &= m+1. && (4.26)
\end{aligned}$$

Note that the existence of fifth zero of (4.22) with $t_5 = 1$ shows that asymptotically seven periodic solution is unstable. □

Remark. *If the initial conditions in* $(-1,0)$ *and* $m \in (-1,0)$ *then Eq.(1.1) has an asymptotically seven periodic solution as*

$$\cdots,0,-1,-1,0,-m-1,-1,m,\cdots.$$

Proof. Firstly we consider from (4.1)-(4.7) and we take t_1, t_2, t_3 or t_4. Therefore,

$$\begin{aligned}
\lim_{n\to\infty} x_{7n-4} &= \lim_{n\to\infty} (0 + a \cdot t^n) = 0, \\
\lim_{n\to\infty} x_{7n-3} &= \lim_{n\to\infty} (-1 + b \cdot t^n) = -1, \\
\lim_{n\to\infty} x_{7n-2} &= \lim_{n\to\infty} (-1 + c \cdot t^n) = -1, \\
\lim_{n\to\infty} x_{7n-1} &= \lim_{n\to\infty} (0 + d \cdot t^n) = 0, \\
\lim_{n\to\infty} x_{7n} &= \lim_{n\to\infty} (-m - 1 + e \cdot t^n) = -m - 1, \\
\lim_{n\to\infty} x_{7n+1} &= \lim_{n\to\infty} (-1 + f \cdot t^n) = -1, \\
\lim_{n\to\infty} x_{7n+2} &= \lim_{n\to\infty} (m + k \cdot t^n) = m.
\end{aligned}$$

Thus, asymptotically seven periodic cycle of Eq.(1.1) has been completed as desired. □

5. Numerical Examples

Now, we present some figures for Eq.(1.1) in order to confirm the above theoretical results.

Example 5.1. *With the initial conditions $x_{-4} = -1, x_{-3} = -1, x_{-2} = 0, x_{-1} = 4$ and $x_0 = -1$, Eq.(1.1) has seven periodic solutions. Moreover, the cycle of these periodic solutions is*

$$(-5, 0, -1, -1, 0, 4, -1, \cdots).$$

Figure 1 verify our theoretical results (see Theorem 3.6).

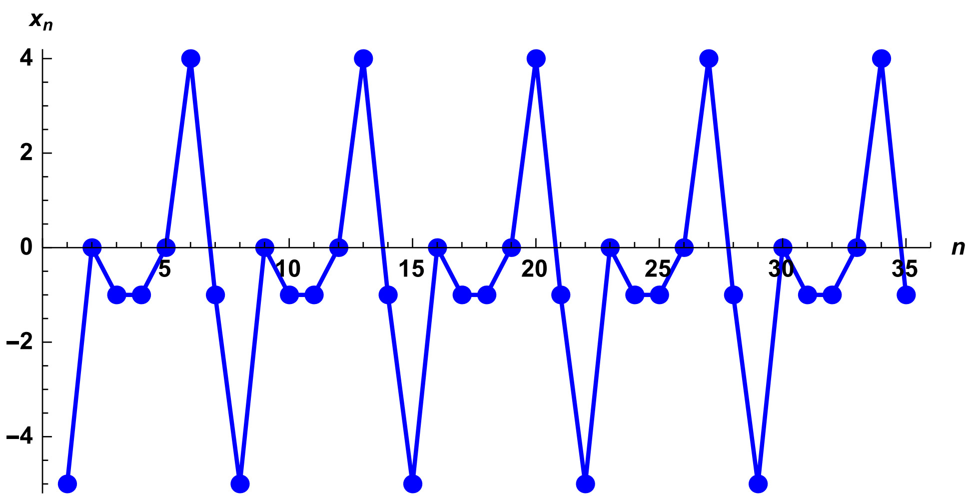

Figure 1. Plot of Eq.(1.1) with the initial conditions $x_{-4} = -1$, $x_{-3} = -1$, $x_{-2} = 0$, $x_{-1} = 4$ and $x_0 = -1$.

Example 5.2. *With the initial conditions $x_{-4} = -0.45$, $x_{-3} = -0.9$, $x_{-2} = -0.2$, $x_{-1} = -0.55$ and $x_0 = -0.35$, Eq.(1.1) has asymptotically seven periodic solutions. Figure 2 verify our theoretical results (Theorem 4.1).*

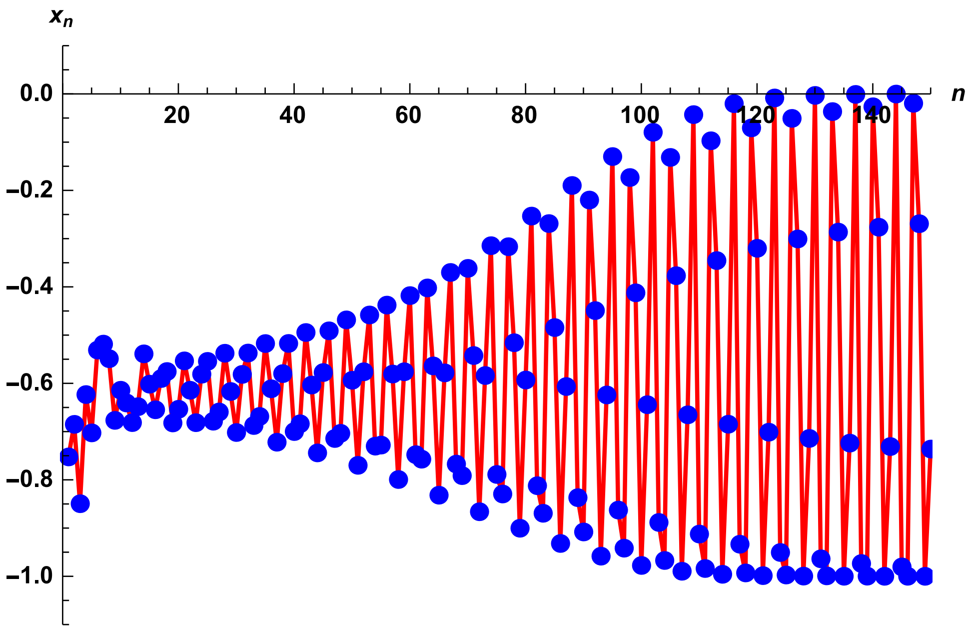

FIGURE 2. Plot of Eq.(1.1) with the initial conditions $x_{-4} = -0.45$, $x_{-3} = -0.9$, $x_{-2} = -0.2$, $x_{-1} = -0.55$ and $x_0 = -0.35$.

6. Conclusion

This study discussed about stability analysis and periodic solutions of Eq.(1.1). In Section 2 we proved that the equilibrium points of Eq.(1.1) are unstable that both are saddle point. In Section 3 we showed that the existence of periodic solutions of Eq.(1.1). In Section 4 we proved that Eq.(1.1) has an asymptotically seven periodic solution. In Section 5 we given two numerical examples so as to confirm our theoretical results.

References

[1] A. Gelişken, C. Cinar, İ. Yalcinkaya, *On the Periodicity of a Difference Equation with Maximum*, Discrete Dyn. Nat. Soc., **2008**, (2008) 1–11. doi:10.1155/2008/820629

[2] C.M. Kent, W. Kosmala, *On the Nature of Solutions of the Difference Equation* $x_{n+1} = x_n x_{n-3} - 1$, International Journal of Nonlinear Analysis and Applications **2** 2 (2011) 24–43.

[3] C.M. Kent, W. Kosmala, M.A. Radin, S. Stević, *Solutions of the difference equation* $x_{n+1} = x_n x_{n-1} - 1$, Abstr. Appl. Anal., **2010** (2010), 1–13. doi:10.1155/2010/469683

[4] C.M. Kent, W. Kosmala, S. Stevic, *Long-term behavior of solutions of the difference equation* $x_{n+1} = x_{n-1} x_{n-2} - 1$, Abstr. Appl. Anal., **2010** (2010), 1–17. doi:10.1155/2010/152378

[5] C.M. Kent, W. Kosmala, S. Stevic, *On the difference equation* $x_{n+1} = x_n x_{n-2} - 1$, Abstr. Appl. Anal., **2011** (2011), 1–15. doi:10.1155/2011/815285

[6] E. Camouzis, G. Ladas**,** *Dynamics of third order rational difference equations with open problems and conjectures, volume 5 of Advances in Discrete Mathematics and Applications,* Chapman & Hall/CRC, Boca Raton, FL, (2008).

[7] E. Taşdemir, Y. Soykan, *On the Periodicies of the Difference Equation* $x_{n+1} = x_n x_{n-1} + \alpha$, Karaelmas Science and Engineering Journal, **6 2** (2016), 329–333.

[8] E. Taşdemir, Y. Soykan, *Long-Term Behavior of Solutions of the Non-Linear Difference Equation* $x_{n+1} = x_{n-1} x_{n-3} - 1$, Gen. Math. Notes, **38 1** (2017), 13–31.

[9] E. Taşdemir, Y. Soykan, *Stability of Negative Equilibrium of a Non-Linear Difference Equation*, J. Math. Sci. Adv. Appl., **49 1** (2018), 51–57.
[10] E. Taşdemir, Y. Soykan, *Dynamical Analysis of a Non-Linear Difference Equation*, J. Comput. Anal. Appl., **26 2** (2019), 288–301.
[11] E.A. Grove, G. Ladas, *Periodicities in nonlinear difference equations (Vol. 4)*. CRC Press, (2004).
[12] E.M. Elsayed, *New method to obtain periodic solutions of period two and three of a rational difference equation*, Nonlinear Dyn., **79 1** (2015), 241–250.
[13] G. Karakostas, *Asymptotic 2–periodic difference equations with diagonally self–invertible responses*, J. Difference Equ. Appl., **6 3** (2000), 329–335.
[14] H. El-metwallya, E.A. Grove, G. Ladas, H.D. Voulov, *On the global attractivity and the periodic character of some difference equations,* J. Difference Equ. Appl., **7 6** (2001), 837–850.
[15] İ. Okumuş, Y. Soykan, *On the Stability of a Nonlinear Difference Equation*, Asian Journal of Mathematics and Computer Research, **17 2** (2017), 88–110.
[16] İ. Okumuş, Y. Soykan, *Some Technique To Show The Boundedness Of Rational Difference Equations*, Journal of Progressive Research in Mathematics, **13 2** (2018), 2246–2258.
[17] İ. Okumuş, Y. Soykan, *Dynamical behavior of a system of three-dimensional nonlinear difference equations,* Adv. Difference Equ., **2018:224** (2018), 1–15.
[18] J. Diblik, M. Feckan, M. Pospisil, *Nonexistence of periodic solutions and S-asymptotically periodic solutions in fractional difference equations,* Appl. Math. Comput., **257** (2015), 230–240.
[19] J. Popenda, E. Schmeidel, *Asymptotically periodic solution of some linear difference equations*, Facta Univ. Ser. Math. Inform., **14** (1999), 31–40
[20] K. Liu, P. Li, F. Han, W. Zhong, *Behavior of the Difference Equations* $x_{n+1} = x_n x_{n-1} - 1$, J. Comput. Anal. Appl., **22 1** (2017), 1361–1370.
[21] L. Berg, *On the asymptotics of nonlinear difference equations*, Z. Anal. Anwend. **21 4** (2002), 1061–1074.
[22] M. Göcen, M. Güneysu, *The global attractivity of some rational difference equations,* J. Comput. Anal. Appl., **25 7** (2018), 1233–1243.
[23] M. Göcen, A. Cebeci, *On the Periodic Solutions of Some Systems of Higher Order Difference Equations*, Rocky Mountain J. Math., **48 3** (2018), 845–858.
[24] M. Gümüş, *Global dynamics of a third-order rational difference equation,* Karaelmas Science and Engineering Journal, **8 2** (2018), 585–589.
[25] M. Gümüş, Y. Soykan, *Global character of a six-dimensional nonlinear system of difference equations*, Discrete Dyn. Nat. Soc., **2016** (2016), 1–7.
[26] R.P. Agarwal, J. Popenda, *Periodic solutions of first order linear difference equations*, Mathl. Comput. Modelling, **22 1** (1995), 11–19.
[27] R.P. Agarwal, P.J. Wong, *Advanced topics in difference equations (Vol. 404)*, Springer Science & Business Media, (2013).
[28] S. Elaydi. *An Introduction to Difference Equations*, Springer-Verlag, New York, Inc, (1996).
[29] S. Elaydi, R.J. Sacker, *Global stability of periodic orbits of non-autonomous difference equations and population biology,* J. Differential Equations, **208 1** (2005), 258–273.
[30] S. Stevic, *Asymptotics of some classes of higher-order difference equations*, Discrete Dyn. Nat. Soc., **2007** (2007), 1–20. doi: http://dx.doi.org/10.1155/2007/56813
[31] S. Stevic, J. Diblik, B. Iricanin, Z. Smarda, Z. *Solvability of nonlinear difference equations of fourth order,* Electron. J. Differential Equations, **264** (2014), 1–14.
[32] S. Stevic, M.A. Alghamdi, A. Alotaibi, *Boundedness character of the recursive sequence*, Appl. Math. Lett., **50** (2015), 83–90.
[33] V.V. Khuong, M.N. Phong, *A Note on the Asymptotic Solution of a Certain Rational Difference Equation*, Int. J. Difference Equ., **6 1** (2011), 53–57.
[34] Y. Wang, Y. Luo, Z. Lu, *Convergence of solutions of* $x_{n+1} = x_n x_{n-1} - 1$, Appl. Math. E-Notes, **12** (2012), 153–157.

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