DEU FMD 22(64), 59-66, 2020

Dokuz Eylül Üniversitesi Mühendislik Fakültesi Fen ve Mühendislik Dergisi
Dokuz Eylul University Faculty of Engineering Journal of Science and Engineering

Basılı/Printed ISSN: 1302-9304. Elektronik/Online ISSN: 2547-958X

The Characterizations of the Spherical Images of Timelike W-Curves in Minkowski Space-Time

Minkowski Uzay-Zamanda Timelike W-Eğrilerin Küresel Göstergelerinin Karakterizasyonları

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Geliş Tarihi /Received: 05.03.2019
Kabul Tarihi / Accepted: 29.07.2019

Araştırma Makalesi/Research Article
DOI: 10.21205/deufmd.2020226407

Atıf şekli/ How to cite: ÜNLÜTÜRK, Y., YILMAZ, S., ÇİMDİKER, M. (2020). The Characterizations of the Spherical Images of Timelike W-Curves in Minkowski Space-Time. DEUFMD, 22(64),59-66.

Abstract

We know that W −curve is a curve which has constant Frenet curvatures. In this study, firstly, we have investigated the principal normal and binormal spherical images of a timelike W −curve on pseudohyperbolic space $\mathbb{H}_0^3$ in Minkowski space-time E_1^4. Besides, the binormal spherical image of the timelike W −curve is a spacelike curve which lies on pseudohyperbolic space $\mathbb{H}_0^3$. Hence, we have obtained the Frenet-Serret invariants of the mentioned image curve in terms of the invariants of the timelike W-curve in the same space. Finally, we have given some characterizations of the spherical image in the case of being helix for the timelike W −curve in Minkowski space-time E_1^4.

Keywords: *Spherical Images, Timelike W-Curve, General Helix, CCR-Curves*

Özet

W −eğrisinin sabit Frenet eğriliklerine sahip bir eğri olduğunu biliyoruz. Bu çalışmada, öncelikle, E_1^4 Minkowski uzay-zamanında, $\mathbb{H}_0^3$ pseudohiperbolik uzay üzerinde bir timelike W −eğrisinin asli normal ve binormal küresel göstergelerini araştırdık. Yanısıra, $\mathbb{H}_0^3$ pseudohiperbolik uzay üzerinde yatan timelike W −eğrisinin binormal küresel göstergesi spacelike bir eğridir. Bu nedenle, aynı uzayda, söz konusu görüntü eğrisinin Frenet-Serret değişmezlerini timelike W −eğrisinin değişmezleri cinsinden elde ettik. Son olarak, E_1^4 Minkowski uzay-zamanındaki timelike W −eğrisi için helis olması durumunda küresel göstergenin bazı karakterizasyonlarını verdik.

Anahtar Kelimeler: *Küresel Göstergeler, Timelike W −Eğrileri, Genel Helis, CCR-Eğriler*

1. Introduction

Lorentzian geometry helps to bridge the gap between modern differential geometry and the mathematical physics of general relativity by giving an invariant treatment of Lorentzian geometry. Nearly a century ago, Einstein's formulation of general relativity expressed in terms of Lorentzian geometry was attractive for geometricians who could penetrate suprisingly into cosmology (redshift, expanding universe, big bang)[1].

Despite its long history, the theory of curve is still one of the most interesting topics in differential geometry and it is still being studied by many mathematicians until now. A tetrad of mutually orthogonal unit vectors (called tangent, normal, binormal, trinormal) was defined and constructed at each point of a differentiable curve. The rates of change of these vectors along the curve define the curvatures of the curve in the four dimensional space. Spherical images of a regular curve in the Euclidean space are obtained by means of Frenet-Serret frame vector fields, so the mentioned topic is a well-known concept in differential geometry of the curves [2]. Also, these kind of curves were studied in four dimensional Euclidean and Lorentzian space [3,4,5,6,7].

W-curve is another curve among the prominent curves which have the constant Frenet curvature. All $W-$curves in Minkowski 3-space are completely classified by Walrave in [3]. Besides, in Minkowski space-time, the spacelike, timelike, null W-curves are studied [8,9].

In this study, we have investigated the principal normal and binormal spherical images of a timelike $W-$curve on pseudohyperbolic space $\mathbb{H}_0^3$ in Minkowski space-time E_1^4. The binormal spherical image of the timelike W-curve is a spacelike curve which lies on pseudohyperbolic space $\mathbb{H}_0^3$. Hence, we have obtained the Frenet-Serret invariants of the mentioned image curve in terms of the invariants of the timelike $W-$curve. Finally, we have given some characterizations of the spherical image in the case of being helix for the timelike $W-$curve in Minkowski space-time E_1^4.

2. Material and Method

Minkowski spacetime E_1^4 is a real vector space R^4 furnished with the standard indefinite flat metric g defined by

$$g = -dx_1 + dx_2 + dx_3 + dx_{4,}$$

where (x_1, x_2, x_3, x_4) is a rectangular coordinate system in E_1^4 [1]. Since g is an indefinite metric, recall that a vector $v \in E_1^4$ can have one of the three causal characters; it can be spacelike if $g(v,v) > 0$ or $v = 0$, timelike if $g(v,v) < 0$ and null (lightlike) if $g(v,v) = 0, v \neq 0$. Similarly, an arbitrary curve $\alpha = \alpha(s)$ in E_1^4 can be locally spacelike, timelike or null (lightlike), if all of its velocity vectors $\alpha'(s)$ are spacelike, timelike or null (lightlike), recpectively. Also, the norm of the vector v is given by

$$\|v\| = \sqrt{|g(v,v)|}.$$

The vector v is a unit vector if $g(v,v) = \mp 1$. Vectors v, w in E_1^4 are said to be orthogonal if $g(v,w) = 0$ [10]. Let u and v be two spacelike vectors in E_1^4, then there is a unique real number $0 \leq \delta \leq \pi$, called the angle between u and v, such that $g(u,v) = \|u\|\|v\| cos\delta$ [11].

The pseudohyperbolic space with the center $m = (m_1, m_2, m_3, m_4) \in E_1^4$ and radius $r \in \mathbb{R}^+$ in the spacetime E_1^4 is the hyperquadric

$$\mathbb{H}_0^3 = \{a = (a_1, a_2, a_3, a_4) \in E_1^4 \mid g(a-m, a-m) = -r^2\},$$

with dimension 3 and index 0 [1].

Let $\varphi = \varphi(s)$ be a curve in E_1^4. If the tangent vector field of this curve forms a constant angle with a constant vector field U, then this curve is called a general helix. Recall that, if a regular curve has constant Frenet-Serret curvatures ratios in E_1^4, then it is called a ccr-curve [12,13,14]. Also, if these curvatures are non-zero constants, the curve is said to be $W-$curve (or helix) [15,16,17].

Denote by $\{T(s), N(s), B_1(s), B_2(s)\}$ the moving Frenet-Serret frame along the curve $\varphi(s)$ in E_1^4. Then T, N, B_1, B_2 are, respectively, the tangent, the principal normal, the binormal (the first binormal) and the trinormal (the second binormal) vector fields. A spacelike or timelike curve $\varphi(s)$ is said to be parametrized by arc-length function s, if $g(\varphi'(s), \varphi'(s)) = \pm 1$.

Let $\varphi(s)$ be a timelike curve in E_1^4, parametrized by arc-length function s. Then the following Frenet-Serret equations are given in [3]:

$$\begin{bmatrix} T' \\ N' \\ B_1' \\ B_2' \end{bmatrix} = \begin{bmatrix} 0 & \kappa & 0 & 0 \\ \kappa & 0 & \tau & 0 \\ 0 & -\tau & 0 & \sigma \\ 0 & 0 & -\sigma & 0 \end{bmatrix} \begin{bmatrix} T \\ N \\ B_1 \\ B_2 \end{bmatrix}, \qquad (1)$$

where T, N, B_1, B_2 are mutually orthogonal vectors satisfying equations

$g(T,T) = -1,$

$g(N,N) = g(B_1,B_1) = g(B_2,B_2) = 1,$

and where κ, τ, σ are the first, second, and third curvatures of the curve φ, respectively.

In the same space, the authors expressed a characterization of spacelike curves lying on $\mathbb{H}_0^3$ by the following theorem:

Theorem 2.1. Let φ(s) be an unit speed spacelike curve in E_1^4, with the spacelike vectors N,B_1 and the curvatures $\kappa \neq 0, \tau \neq 0$, $\sigma \neq 0$ for each $s \in I \subset \mathbb{R}$. Then, the curve φ lies on pseudohyperbolic space if and only if

$$\frac{\sigma}{\tau}\frac{d\rho}{ds} = \frac{d}{ds}\left[\frac{1}{\sigma}\left(\rho\tau + \frac{d}{ds}\left(\frac{1}{\tau}\frac{d\rho}{ds}\right)\right)\right], \qquad (2)$$

where

$$\left\{\frac{1}{\sigma}\left(\rho\tau + \frac{d}{ds}\left(\frac{1}{\tau}\frac{d\rho}{ds}\right)\right)\right\}^2 > \rho^2 + \left(\frac{1}{\tau}\frac{d\rho}{ds}\right)^2$$

and $\rho = \frac{1}{\kappa}$ [15].

Definition 2.2. Let $a = (a_1, a_2, a_3, a_4), b = (b_1, b_2, b_3, b_4)$ and $c = (c_1, c_2, c_3, c_4)$ be vectors in E_1^4. The vector product is defined by

$$a \times b \times c = -\begin{vmatrix} -e_1 & e_2 & e_3 & e_4 \\ a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \\ c_1 & c_2 & c_3 & c_4 \end{vmatrix},$$

where e_1, e_2, e_3, e_4 are mutually orthogonal vectors (coordinate direction vectors) satisfying equations

$e_1 \times e_2 \times e_3 = e_4, \qquad e_2 \times e_3 \times e_4 = e_1,$

$e_3 \times e_4 \times e_1 = e_2, \qquad e_4 \times e_1 \times e_2 = -e_3,$

[5].

Theorem 2.3. Let φ(s) be an arbitrary spacelike curve in E_1^4. The Frenet-Serret apparatus of the curve φ can be written as follows:

$$\begin{aligned} T &= \frac{\varphi'}{\|\varphi'\|}, \\ N &= \frac{\|\varphi'\|^2\varphi'' - g(\varphi',\varphi'')\varphi'}{\|\|\varphi'\|^2\varphi'' - g(\varphi',\varphi'')\varphi'\|}, \\ B_1 &= \mu N \times T \times B_2, \\ B_2 &= \mu\frac{T \times N \times \varphi'''}{\|T \times N \times \varphi'''\|}, \end{aligned} \qquad (3)$$

and

$$\begin{aligned} \kappa &= \frac{\left\|\|\varphi'\|^2\varphi'' - g(\varphi',\varphi'')\varphi'\right\|}{\|\varphi'\|^4}, \\ \tau &= \frac{\|T \times N \times \varphi'''\|\|\varphi'\|}{\|\|\varphi'\|^2\varphi'' - g(\varphi',\varphi'')\varphi'\|}, \\ \sigma &= \frac{g(\varphi^{IV}, B_2)}{\|T \times N \times \varphi'''\|\|\varphi'\|}, \end{aligned} \qquad (4)$$

where μ is taken -1 or 1 to make 1 the determinant of $\{T, N, B_1, B_2\}$ matrix [5].

3. Results

3.1. The principal normal spherical image of a timelike W −curve in E_1^4

In this section, we give the definition of the principal normal spherical image for the timelike W −curves in Minkowsk space-time E_1^4.

Definition 3.1. Let $\beta = \beta(s)$ be a unit speed timelike W −curve in Minkowski space-time E_1^4. If we translate the principal normal vector to the center O of the pseudohyperbolic space $\mathbb{H}_0^3$, we obtain a curve $\delta = \delta(s_\delta)$. This curve is called the principal normal spherical indicatrix or image of the curve β in E_1^4.

Theorem 3.2. Let $\beta = \beta(s)$ be a unit speed timelike W −curve and $\delta = \delta(s_\delta)$ be its principal normal spherical image. Then,

i) $\delta = \delta(s_\delta)$ is a spacelike curve if the first and second curvatures of $\beta(s)$ satisfy the following

$$\tau < \kappa < 0, \qquad 0 < \kappa < \tau.$$

ii) Frenet-Serret apparatus of the curve $\delta, \{T_\delta, N_\delta, B_{1\delta}, B_{2\delta}, \kappa_\delta, \tau_\delta, \sigma_\delta\}$ can be formed by the apparatus of the curve β.

Proof. Let $\beta = \beta(s)$ be a unit speed timelike W-curve and $\delta = \delta(s_\delta)$ be its principal normal spherical indicatrix. It can be written as

$$\delta = N(s). \tag{5}$$

Differentiating the equation (5) with respect to s, then we obtain

$$\delta' = \dot{\delta}\frac{ds_\delta}{ds} = \kappa T + \tau B_1. \tag{6}$$

Here, we shall denote differentiation according to s by a dash, and differentiation according to s_δ by a dot. Thus, we obtain the unit tangent vector of the principal normal spherical image curve δ as

$$T_\delta = \frac{\kappa T + \tau B_1}{\sqrt{\tau^2 - \kappa^2}}, \tag{7}$$

and

$$\|\delta'\| = \frac{ds_\delta}{ds} = \sqrt{\tau^2 - \kappa^2}. \tag{8}$$

The causal character of the principal normal spherical image curve δ is determined by the following inner product:

$$g(\delta', \delta') = \tau^2 - \kappa^2. \tag{9}$$

From the expression (9), we will take the spherical image curve as spacelike one by assuming the conditions

$$\tau < \kappa < 0, \qquad 0 < \kappa < \tau. \tag{10}$$

Considering the previous method and using the property of the curve to be W −curve, we form the following differentiations with respect to s:

$$\begin{aligned} \delta'' &= (\kappa^2 - \tau^2)N + \tau\sigma B_2, \\ \delta''' &= \kappa(\kappa^2 - \tau^2)T + \tau(\kappa^2 - \tau^2 - \sigma^2)B_1, \\ \delta^{(IV)} &= ((\kappa^2 - \tau^2)^2 + \tau^2\sigma^2)N + \tau\sigma(\kappa^2 - \tau^2 - \sigma^2)B_2. \end{aligned} \tag{11}$$

By the expressions (2), we arrive at

$$\|\delta'\|^2\delta'' - g(\delta', \delta'')\delta' = -(\kappa^2 - \tau^2)^2 N + \tau\sigma(\tau^2 - \kappa^2)B_2. \tag{12}$$

Then, we can write the principal normal vector of the spherical image curve δ

$$N_\delta = \frac{\kappa^2 - \tau^2}{\sqrt{(\tau\sigma)^2 + (\tau^2 - \kappa^2)^2}} N + \frac{\tau\sigma}{\sqrt{(\tau\sigma)^2 + (\tau^2 - \kappa^2)^2}}.B_2, \tag{13}$$

and the first curvature of the spherical image curve δ is obtained by

$$\kappa_\delta = \frac{\sqrt{(\tau\sigma)^2 + (\tau^2 - \kappa^2)^2}}{\tau^2 - \kappa^2}. \tag{14}$$

Now, calculate the vector product

$U = T_\delta \times N_\delta \times \delta'''$, that is, we have the vector U as

$$U = \frac{-\kappa\tau\sigma^2}{\sqrt{\tau^2 - \kappa^2}}\left(\frac{-\tau\sigma}{\sqrt{(\tau\sigma)^2 + (\tau^2 - \kappa^2)^2}}N + \frac{\kappa^2 - \tau^2}{\sqrt{(\tau\sigma)^2 + (\tau^2 - \kappa^2)^2}}B_2\right). \tag{15}$$

Hence, we obtain the trinormal (second binormal) vector field of the principal normal spherical image curve δ as follows:

$$B_{2\delta} = \mu\left(\frac{\tau\sigma}{\sqrt{(\tau\sigma)^2 + (\tau^2 - \kappa^2)^2}}N + \frac{\tau^2 - \kappa^2}{\sqrt{(\tau\sigma)^2 + (\tau^2 - \kappa^2)^2}}B_2\right). \tag{16}$$

Taking the norm of both sides of the expressions (15) and (12) then the second curvature of the principal normal spherical image curve δ is

$$\tau_\delta = \frac{-\kappa\tau\sigma^2}{(\tau^2 - \kappa^2)\sqrt{(\tau\sigma)^2 + (\tau^2 - \kappa^2)^2}}. \tag{17}$$

To obtain the binormal vector field of the principal normal spherical image curve δ,we express $V = N_\delta \times T_\delta \times B_{2\delta}$ as follows:

$$V = -\frac{\tau}{\sqrt{\tau^2 - \kappa^2}}T - \frac{\kappa}{\sqrt{\tau^2 - \kappa^2}}B_1. \tag{18}$$

From the expression (18), then we get the binormal vector of the principal normal spherical image curve δ

$$B_{1\delta} = \mu\left(-\frac{\tau}{\sqrt{\tau^2-\kappa^2}}T - \frac{\kappa}{\sqrt{\tau^2-\kappa^2}}B_1\right) \quad (19)$$

Using the equation (16), the third curvature is given by

$$\sigma_\delta = \mu\frac{\kappa\sigma}{\sqrt{(\tau\sigma)^2+(\tau^2-\kappa^2)^2}}. \quad (20)$$

Corollary 3.3. Frenet-Serret apparatus of the principal normal spherical image curve δ is an orthonormal frame of Minkowski space-time E_1^4.

Proof. It can be straightforwardly seen by using the equations (7), (13), (16), (19).

Corollary 3.4. Let $\beta = \beta(s)$ be a unit speed timelike W −curve and $\delta = \delta(s_\delta)$ be its principal normal spherical image. Then, the curve δ is also a helix.

Proof. Let $\beta = \beta(s)$ be a unit speed timelike W −curve. We know that the curvature functions are constants. Therefore, we know that the curvature functions of the principal normal spherical image $\delta(s_\delta)$ are constants by means of the equations (14), (17) and (20). Hence, the curve $\delta(s_\delta)$ becomes W −curve which is the special case of helix.

Theorem 3.5. Let $\beta = \beta(s)$ be a unit speed timelike W −curve and $\delta = \delta(s_\delta)$ be its principal normal spherical image. If δ is a general helix, then its fixed direction Φ is composed

$$\begin{aligned}\Phi &= \left(-\frac{1}{2}x_1\kappa s^2 - x_2\kappa s + x_3\right)T \\ &+(x_1 s + x_2)N \\ &+\left(\begin{array}{c}-\frac{1}{2\tau}x_1\kappa^2 s^2 - \frac{1}{\tau}x_2\kappa^2 s \\ +\ \frac{1}{\tau}x_3\kappa + \frac{x_1}{\tau}\end{array}\right)B_1 \quad (21) \\ &+\left(\begin{array}{c}\frac{1}{6\tau}x_1\kappa^2\sigma s^3 + \frac{1}{2\tau}x_2\kappa^2\sigma s^2 \\ -\frac{1}{\tau}x_3\kappa\sigma - \frac{x_1\sigma}{\tau}s + x_4\end{array}\right)B_2,\end{aligned}$$

where x_1 is a non-zero constant and x_2, x_3, x_4 are constants.

Proof. Let $\beta = \beta(s)$ be a unit speed timelike W −curve and $\delta = \delta(s_\delta)$ be its principal normal spherical image. If δ is a general helix, then for a spacelike vector Φ, we may express

$$g(T_\delta, \Phi) = \cos\theta, \quad (22)$$

where θ is a constant angle. The equation (22) is also congruent to

$$g\left(\frac{\kappa T + \tau B_1}{\sqrt{\tau^2-\kappa^2}}, \Phi\right) = \cos\theta. \quad (23)$$

The constant vector Φ according to $\{T, N, B_1, B_2\}$ is formed as

$$\Phi = \varepsilon_1 T + \varepsilon_2 N + \varepsilon_3 B_1 + \varepsilon_4 B_2 \quad (24)$$

Differentiating the expression (24) with respect to s, then we have the following system of ordinary differential equations

$$\begin{cases}\varepsilon_1' + \varepsilon_2\kappa = 0 \\ \varepsilon_1\kappa + \varepsilon_2' - \varepsilon_3\tau = 0 \\ \varepsilon_2\tau - \varepsilon_4\sigma + \varepsilon_3' = 0 \\ \varepsilon_4' + \varepsilon_3\sigma = 0\end{cases} \quad (25)$$

We know that $-\varepsilon_1\kappa + \varepsilon_3\tau = x_1 \neq 0$ is a non-zero constant. Since the curve $\beta = \beta(s)$ is a W −curve, its curvature functions are constants. Then the solution of the system (25) can be obtained as

$$\begin{aligned}\varepsilon_1 &= -\frac{1}{2}x_1\kappa s^2 - x_2\kappa s + x_3, \\ \varepsilon_2 &= x_1 s + x_2, \\ \varepsilon_3 &= -\frac{1}{2\tau}x_1\kappa^2 s^2 - \frac{1}{\tau}x_2\kappa^2 s \\ &\quad + \frac{1}{\tau}x_3\kappa + \frac{x_1}{\tau}, \\ \varepsilon_4 &= \frac{1}{6\tau}x_1\kappa^2\sigma s^3 + \frac{1}{2\tau}x_2\kappa^2\sigma s^2 \\ &\quad -\frac{1}{\tau}x_3\kappa\sigma s - \frac{x_1\sigma}{\tau}s + x_4.\end{aligned} \quad (26)$$

3.2. The binormal spherical image of a timelike W −curve in E_1^4

In this section, we give the definition of the binormal spherical image for timelike W −curves in Minkowski space-time E_1^4.

Definition 3.6. Let $\beta = \beta(s)$ be a unit speed timelike W −curve in Minkowski space-time E_1^4. If we translate the binormal vector to the center O of the pseudohyperbolic space $\mathbb{H}_0^3$, we obtain a curve $\varphi = \varphi(s_\varphi)$. This curve is called the binormal spherical indicatrix or image of the curve β in E_1^4.

Theorem 3.7. Let $\beta = \beta(s)$ be a unit speed timelike W −curve and $\varphi = \varphi(s_\varphi)$ be its binormal spherical image. Then,

i) $\varphi = \varphi(s_\varphi)$ is a spacelike curve.

ii) Frenet-Serret apparatus of the curve $\varphi, \{T_\varphi, N_\varphi, B_{1\varphi}, B_{2\varphi}, \kappa_\varphi, \tau_\varphi, \sigma_\varphi\}$ can be formed by the apparatus of the curve β.

iii) $\varphi = \varphi(s_\varphi)$ is also a helix lying on the pseudohyperbolic sphere $\mathbb{H}_0^3$ in E_1^4.

Proof. Let $\beta = \beta(s)$ be a unit speed timelike W −curve and $\varphi = \varphi(s_\varphi)$ be its binormal spherical image. It can be written as

$$\varphi = B_1(s). \tag{27}$$

Differentiating the equation (27) with respect to s, then we obtain

$$\varphi' = \dot{\varphi}\frac{ds_\varphi}{ds} = -\tau N + \sigma B_2. \tag{28}$$

Here, we shall denote differentiation according to s by a dash, and differentiation according to s_φ by a dot. Thus, we obtain the unit tangent vector of the binormal spherical image curve φ as

$$T_\varphi = \frac{-\tau N + \sigma B_2}{\sqrt{\tau^2 + \sigma^2}}, \tag{29}$$

and

$$\|\varphi'\| = \frac{ds_\varphi}{ds} = \sqrt{\tau^2 + \sigma^2}. \tag{30}$$

The causal character of the binormal spherical image curve φ is determined by the following inner product:

$$g(\varphi', \varphi') = \tau^2 + \sigma^2. \tag{31}$$

According to the expression (31), the binormal spherical image is a spacelike curve.

Considering the previous method and using the property of the curve to be W −curve, the following differentiations with respect to s are formed:

$$\begin{aligned}
\varphi'' &= -\tau\kappa T - (\tau^2 + \sigma^2)B_1, \\
\varphi''' &= \tau\begin{pmatrix} \tau^2 + \sigma^2 \\ -\kappa^2 \end{pmatrix} N - \sigma(\tau^2 + \sigma^2)B_2, \\
\varphi^{(IV)} &= \tau(\kappa(\tau^2 + \sigma^2) - \kappa^3)T \\
&\quad + ((\tau^2 + \sigma^2)^2 - \tau^2\kappa^2)B_1.
\end{aligned} \tag{32}$$

By the expressions (2), then we get

$$\|\varphi'\|^2\varphi'' - g(\varphi', \varphi'')\varphi' = -(\tau^2 + \sigma^2)\tau\kappa T - (\tau^2 + \sigma^2)^2 B_1. \tag{33}$$

Then, we can get the principal normal vector of the binormal spherical image curve φ

$$\begin{aligned}
N_\varphi = &-\frac{\kappa\tau}{\sqrt{|(\tau^2 + \sigma^2)^2 - (\tau\kappa)^2|}}T \\
&-\frac{\tau^2 + \sigma^2}{\sqrt{|(\tau^2 + \sigma^2)^2 - (\tau\kappa)^2|}}B_1,
\end{aligned} \tag{34}$$

and the first curvature of the binormal spherical image curve φ is as:

$$\kappa_\varphi = \frac{\sqrt{|(\tau^2 + \sigma^2)^2 - (\tau\kappa)^2|}}{\tau^2 + \sigma^2}. \tag{35}$$

The vector product $X = T_\varphi \times N_\varphi \times \varphi'''$ is given by

$$X = -\frac{1}{\sqrt{|(\tau^2 + \sigma^2)^2 - (\tau\kappa)^2|(\tau^2 + \sigma^2)}}\begin{pmatrix} (\tau^2 + \sigma^2)T \\ +(\kappa\tau)B_1 \end{pmatrix}.$$

(36)

Using the expression (36), then the trinormal (second binormal) vector field of the binormal spherical image curve φ is obtained as

$$B_{2\varphi} = -\frac{\mu}{\kappa^2\tau\sigma\sqrt{|(\tau^2 + \sigma^2)^2 - (\tau\kappa)^2|}}\begin{pmatrix} (\tau^2 + \sigma^2)T \\ +(\kappa\tau)B_1 \end{pmatrix}. \tag{37}$$

Taking the norm of both sides of the equations (33) and (36), then we find the second curvature of the binormal spherical image curve φ

$$\tau_\varphi = \frac{\kappa^2\tau\sigma}{(\tau^2 + \sigma^2)\sqrt{(\tau^2 + \sigma^2)^2 - (\kappa\tau)^2}}. \tag{38}$$

The binormal vector field of the the binormal spherical image curve φ is expressed as

$$B_{1\varphi} = -\frac{\mu}{\sqrt{\tau^2-\kappa^2}}\begin{pmatrix}\sigma N\\ +\tau B_2\end{pmatrix}. \tag{39}$$

Finally, using the equation (39), then the third curvature of the the binormal spherical image curve φ is obtained by

$$\sigma_{\varphi} = 0. \tag{40}$$

Corollary 3.8. Frenet-Serret apparatus of the binormal spherical image φ is an orthonormal frame of Minkowski space-time E_1^4.

Proof. It can be straightforwardly seen by using the equations (29), (34), (37), (39).

Corollary 3.9. Let $\beta = \beta(s)$ be a unit speed timelike $W-$curve and $\varphi = \varphi(s_{\varphi})$ be its binormal spherical image. Then, the curve φ is also a helix.

Proof. Let $\beta = \beta(s)$ be a unit speed timelike $W-$curve. We know that the curvature functions are constants. We know that the curvature functions of the binormal spherical image $\varphi(s_{\varphi})$ are constants. Hence, the curve $\varphi(s_{\varphi})$ becomes $W-$curve which is the special case of helix.

Theorem 3.10. Let $\beta = \beta(s)$ be a unit speed timelike $W-$curve and $\varphi = \varphi(s_{\varphi})$ be its binormal spherical image. If φ is a general helix, then its fixed direction Φ is composed

$$\begin{aligned}\Phi = &\left(\frac{1}{6\tau}x_1\sigma^2\kappa s^3 + \frac{x_2\sigma^2\kappa s^2}{2\tau} - \frac{1}{\tau}x_3\kappa\sigma + \frac{1}{\tau}x_1\kappa\sigma + x_4\right)T\\ &+\left(-\frac{1}{2\tau}x_1\sigma^2 s^2 - \frac{1}{\tau}x_2\sigma^2 s + \frac{1}{\tau}x_3\sigma - \frac{x_1}{\tau}\right)N\\ &+(x_1 s + x_2)B_1\\ &+\left(-\frac{1}{2}x_1\sigma s^2 - x_2\sigma s + x_3\right)B_2,\end{aligned} \tag{41}$$

where x_1 is a non-zero constant and x_2, x_3, x_4 are constants.

Proof. Let $\beta = \beta(s)$ be a unit speed timelike $W-$curve and $\varphi = \varphi(s_{\varphi})$ be its binormal spherical indicatrix. If φ is a general helix, then for a constant spacelike vector Φ, we may express

$$g(T_{\varphi}, \Phi) = \cos\theta, \tag{42}$$

where θ is a constant angle. The equation (28) is also congruent to

$$g\left(\frac{-\tau N + \sigma B_2}{\sqrt{\tau^2+\sigma^2}}, \Phi\right) = \cos\theta. \tag{43}$$

The constant vector Φ according to $\{T, N, B_1, B_2\}$ is formed as

$$\Phi = \varepsilon_1 T + \varepsilon_2 N + \varepsilon_3 B_1 + \varepsilon_4 B_2 \tag{44}$$

Differentiating the expression (43) with respect to s, then we have the following system of ordinary differential equations

$$\begin{cases}\varepsilon_1' + \varepsilon_2\kappa = 0\\ \varepsilon_1\kappa + \varepsilon_2' - \varepsilon_3\tau = 0\\ \varepsilon_2\tau - \varepsilon_4\sigma + \varepsilon_3' = 0\\ \varepsilon_4' + \varepsilon_3\sigma = 0\end{cases} \tag{45}$$

We know that $-\varepsilon_2\tau + \varepsilon_4\sigma = x_1 \neq 0$ is a non-zero constant. Since the curve $\beta = \beta(s)$ is a $W-$curve, its curvature functions are constants. Then the solution of the system (44) can be obtained as

$$\begin{aligned}\varepsilon_1 &= \frac{1}{6\tau}x_1\sigma^2\kappa s^3 + \frac{x_2\sigma^2\kappa s^2}{2\tau} - \frac{1}{\tau}x_3\kappa\sigma + \frac{1}{\tau}x_1\kappa\sigma + x_4,\\ \varepsilon_2 &= -\frac{1}{2\tau}x_1\sigma^2 s^2 - \frac{1}{\tau}x_2\sigma^2 s + \frac{1}{\tau}x_3\sigma - \frac{x_1}{\tau},\\ \varepsilon_3 &= x_1 s + x_2,\\ \varepsilon_4 &= -\frac{1}{2}x_1\sigma s^2 - x_2\sigma s + x_3.\end{aligned} \tag{46}$$

4. Discussion and Conclusion

In the present work, we extend spherical image concept to timelike $W-$curve in Minkowski space-time. We investigate principal normal and binormal spherical images of a timelike W −curve and observe that principal normal spherical curves are spacelike curves under certain conditions, and also binormal spherical images occur entirely as spacelike curves. Thereafter, we determine relations between Frenet-Serret invariants of the base curve and its spherical images.

Acknowledgement

The authors would like to thank anonymous reviewers for their valuable comments and suggestions to improve the quality of the paper.

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