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NEW CHARACTERIZATIONS OF SPACELIKE CURVES ON TIMELIKE SURFACES THROUGH THE LINK OF SPECIFIC FRAMES

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Abstract. In this work, considering a regular spacelike curve on a smooth timelike surface in Minkowski 3-space, we investigate relations between the mentioned curve's Darboux and Bishop frames on the timelike surface. Next we obtain Darboux vector of the regular spacelike curve in terms of Bishop apparatus. Thereafter, translating the Darboux vector to the center of the unit sphere, we determine aforementioned spacelike curve. Moreover, we investigate this spherical image's Frenet-Serret and Bishop apparatus and illustrate our results with two examples.

Keywords: Spacelike curve, Darboux frame, Darboux vector, Type-2 Bishop frame.

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1. Introduction

The local theory of space curves are mainly developed by the Frenet-Serret formulas. This way approaching the curves opened a door to a moving frame for regular curves which was expressed by the derivative formulas of a chosen basis of the Euclidean space. Thereafter by the help of specific ordinary differential equations, further classical topics such as spherical images, involute-evolute curves, Bertrand curves, etc. were treated by the researchers, for details see [2, 7, 9].

After the development on the local theory, researchers aimed to generalize the obtained concepts to the theory of surfaces. For a regular curve on a smooth surface, it is a well-known concept that, Darboux-Ribaucour frame can be constructed. And relations among Frenet-Serret and Darboux frames are mainly presented in [1, 2].Studying relations between some invariants such as geodesic curvature, normal curvature and geodesic torsion for some curves fully lying on a surface in the Minkowski 3-space E_1^3 is a prevalent topic among mathematicians [8, 17, 18].

Bishop frame, which is also called alternative or parallel frame of the curves, was introduced by L.R. Bishop in 1975 by means of parallel vector fields.[3]. Recently, many research papers related to classical differential geometry topics have been done in the Euclidean space, see [6]; in Minkowski space, see [4, 5, 11], and in dual space, see [12]. Also a new version of Bishop frame and its related applications were given in [20].

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In this study, we aim to determine relations among Darboux and Bishop frames of a spacelike curve on a regular timelike surface in Minkowski 3-space E_1^3. We also obtain some results about the relations between the instantaneous vector according to Darboux and Bishop frames for the given curve-surface pair. Then, translating the Darboux vector to the center of the unit sphere, we characterize its spherical image in Minkowski 3-space E_1^3. Further some relations among its Frenet-Serret invariants and Bishop invariants are presented in Minkowski 3-space E_1^3. Also, two examples of the main results are given and their graphs are plotted by aid of a software programme.

2. Preliminaries

The fundamentals of Minkowski 3-space below is cited from [13, 14].

The three dimensional Minkowski space E_1^3 is a real vector space R^3 endowed with the standard flat Lorentzian metric given by

$$\langle , \rangle_L = -dx_1^2 + dx_2^2 + dx_3^2$$

where (x_1, x_2, x_3) is a rectangular coordinate system of E_1^3. This metric is an indefinite one.

Let $u = (u_1, u_2, u_3)$ and $v = (v_1, v_2, v_3)$ be arbitrary vectors in E_1^3, the Lorentzian cross product of u and v is defined as

$$u \times v = -\det \begin{bmatrix} -i & j & k \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{bmatrix}$$

Recall that a vector $v \in E_1^3$ can have one of three Lorentzian characters: it is a spacelike vector if $\langle v, v\rangle > 0$ or $v = 0$; timelike $\langle v, v\rangle < 0$ and null (lightlike) $\langle v, v\rangle = 0$ for $v \neq 0$. Similarly, an arbitrary curve $\alpha = \alpha(s)$ in E_1^3 can locally be spacelike, timelike or null (lightlike) if its velocity vector α' are spacelike, timelike or null (lightlike), respectively, for every $s \in I \subset \mathbb{R}$. The pseudo-norm of an arbitrary vector $a \in E_1^3$ is given by $\|a\| = \sqrt{|\langle a, a\rangle|}$. The curve $\alpha = \alpha(s)$ is called a unit speed curve if its velocity vector α' is a unit one i.e., $\|\alpha'\| = 1$. For vectors $v, w \in E_1^3$, they are said to be orthogonal eachother if and only if $\langle v, w\rangle = 0$.

The angle between two vectors in Minkowski space is defined by [15].

Definition 2.1 i) *Spacelike angle*: Let $\mathbf{x}$ and $\mathbf{y}$ be spacelike vectors in E_1^3 that span a spacelike vector subspace; then we have $|\langle \mathbf{x}, \mathbf{y}\rangle| \leq \|\mathbf{x}\| \|\mathbf{y}\|$, and hence, there is a unique real number $\theta \geq 0$ such that $\langle \mathbf{x}, \mathbf{y}\rangle = \|\mathbf{x}\| \|\mathbf{y}\| \cos\theta$. This number is called the spacelike angle between the vectors $\mathbf{x}$ and $\mathbf{y}$.

ii) *Central angle*: Let $\mathbf{x}$ and $\mathbf{y}$ be spacelike vectors in E_1^3 that span a timelike vector subspace; then we have $|\langle \mathbf{x}, \mathbf{y}\rangle| > \|\mathbf{x}\| \|\mathbf{y}\|$, and hence, there is a unique real number $\theta \geq 0$ such that $\langle \mathbf{x}, \mathbf{y}\rangle = \|\mathbf{x}\| \|\mathbf{y}\| \cosh\theta$. This number is called the central angle between the vectors $\mathbf{x}$ and $\mathbf{y}$.

iii) *Lorentzian timelike angle:* Let $\mathbf{x}$ be spacelike vector and $\mathbf{y}$ be timelike vector in E_1^3. Then there is a unique real number $\theta \geq 0$ such that $\langle \mathbf{x}, \mathbf{y}\rangle = \|\mathbf{x}\| \|\mathbf{y}\| \sinh\theta$. This number is called the Lorentzian timelike angle between the vectors $\mathbf{x}$ and $\mathbf{y}$.

The Lorentzian sphere S_1^2 of radius $r > 0$ and with the center in the origin of the space E_1^3 is defined by

$$S_1^2(r) = \{p = (p_1, p_2, p_3) \in E_1^3 : \langle p, p\rangle = r^2\}.$$

The pseudo-hyperbolic space H_0^2 of radius $r > 0$ and with the center in the origin of the space E_1^3 is defined by

$$H_0^2(r) = \{p = (p_1, p_2, p_3) \in E_1^3 : \langle p, p\rangle = -r^2\}.$$

Denote the moving Serret-Frenet frame along the curve $\alpha = \alpha(s)$ by $\{T, n, B\}$ in the space E_1^3. For an arbitrary spacelike curve $\alpha = \alpha(s)$ in E_1^3, the Serret-Frenet formulae are given as follows

$$\begin{bmatrix} T' \\ n' \\ B' \end{bmatrix} = \begin{bmatrix} 0 & \rho & 0 \\ -\rho & 0 & \tau \\ 0 & \tau & 0 \end{bmatrix} . \begin{bmatrix} T \\ n \\ B \end{bmatrix} \tag{1}$$

where the functions ρ and τ are given, respectively, the first and second (torsion) curvatures as

$$\rho(s) = \sqrt{\langle T'(s), T'(s) \rangle} \text{ and } \tau(s) = -\langle n'(s), B(s) \rangle ,$$

and

$$T(s) = \alpha'(s), n(s) = \frac{T'(s)}{\rho(s)}, B(s) = T(s) \times n(s) \text{ and } \tau(s) = \frac{\det(\alpha', \alpha'', \alpha''')}{\rho^2(s)}.$$

Let $\varphi(u, v)$ be an oriented surface in Minkowski 3-space E_1^3 and let consider a non-null curve $\alpha(s)$ lying on φ fully. Since the curve $\alpha(s)$ lies on the surface φ there exists another frame of the curve $\alpha(s)$ which is called Darboux frame and denoted by $\{T, g, N\}$. In this frame T is the unit tangent of the curve, N is the unit normal of the surface φ and g is a unit vector given by $g = n \times T$. Since the unit tangent T is common in both Frenet frame and Darboux frame, the vectors n, B, g and N lie on the same plane. Then, if the surface φ is an oriented timelike surface, and the curve lying on φ is spacelike, then the derivative formulae of Darboux frame of $\alpha(s)$ is given by

$$\begin{bmatrix} T'(s) \\ g'(s) \\ N'(s) \end{bmatrix} = \begin{bmatrix} 0 & \rho_g & -\rho_n \\ \rho_g & 0 & \tau_g \\ \rho_n & \tau_g & 0 \end{bmatrix} \begin{bmatrix} T(s) \\ g(s) \\ N(s) \end{bmatrix}, \tag{2}$$

where ρ_g, ρ_n and τ_g are, respectively, called the geodesic curvature, the normal curvature, and the geodesic torsion. These functions can be given in terms of ρ and τ as follows

$$\rho_g = \rho \cosh\theta, \quad \rho_n = \rho \sinh\theta, \quad \tau_g = \tau + \frac{d\theta}{ds}.$$

if the surface $\varphi(u, v)$ is timelike and the curve $\alpha(s)$ is spacelike. Also there are the relations between Frenet and Darboux frames as given in the following matrix form

$$\begin{bmatrix} T(s) \\ g(s) \\ N(s) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cosh\theta & \sinh\theta \\ 0 & \sinh\theta & \cosh\theta \end{bmatrix} \begin{bmatrix} T(s) \\ n(s) \\ B(s) \end{bmatrix} \tag{3}$$

where θ is the angle between the vectors g and n [16].

The derivative formulae of type-2 Bishop frame of a spacelike curve with spacelike principal normal are given

$$\begin{bmatrix} \Omega_1'(s) \\ \Omega_2'(s) \\ B'(s) \end{bmatrix} = \begin{bmatrix} 0 & 0 & \xi_1 \\ 0 & 0 & -\xi_2 \\ -\xi_1 & -\xi_2 & 0 \end{bmatrix} . \begin{bmatrix} \Omega_1(s) \\ \Omega_2(s) \\ B(s) \end{bmatrix} \tag{4}$$

in E_1^3. Also, the relations between Frenet and type-2 Bishop frame are given as

$$\begin{bmatrix} T(s) \\ n(s) \\ B(s) \end{bmatrix} = \begin{bmatrix} \sinh\overline{\theta}(s) & -\cosh\overline{\theta}(s) & 0 \\ \cosh\overline{\theta}(s) & -\sinh\overline{\theta}(s) & 0 \\ 0 & 0 & 1 \end{bmatrix} . \begin{bmatrix} \Omega_1(s) \\ \Omega_2(s) \\ B(s) \end{bmatrix}, \tag{5}$$

where the angle $\overline{\theta} = \arctan h\dfrac{\xi_2}{\xi_1}$. And also there are the following expressions

$$\kappa(s)=\overline{\theta}'(s), \quad \tau=\sqrt{\left|\xi_2^2-\xi_1^2\right|}, \quad \xi_1=\tau(s)\cosh\overline{\theta}(s), \quad \xi_2=\tau(s)\sinh\overline{\theta}(s), \tag{6}$$

where $\kappa(s)$ and $\tau(s)$ are the curvature and torsion functions of the curve $\alpha(s)$ and ξ_1, ξ_2 are the curvatures along the vectors Ω_1 and Ω_2, respectively [19]

Instantaneous (Darboux) vector of the Frenet-Serret trihedron is defined by the following equations

$$T' = w \wedge T, \quad n' = w \wedge n, \quad B' = w \wedge B. \tag{7}$$

The Darboux vector of the curve $\alpha(s)$ according to Frenet frame is [16] given as

$$w = -\tau T + \rho B. \tag{8}$$

The Darboux vector of the curve $\alpha(s)$ according to Darboux frame is [16] given as

$$w = -\tau_g T - \rho_n g + \rho_g N. \tag{9}$$

The Darboux vector of the curve $\alpha(s)$ according to Bishop frame of type-1 is given [4] as

$$\overline{w} = k_2 M_1 + k_1 M_2, \tag{10}$$

and also according to type-2 Bishop frame, the vector $\overline{w}$ is given, [21] as

$$\overline{w} = \xi_2 \Omega_1. \tag{11}$$

Let w be unit Darboux vector of a curve on the timelike surface in E_1^3 . If the Darboux vector is translated to the center of the unit sphere, we have a spherical image. This curve is called Darboux spherical image of the curve in in E_1^3. Let $\alpha(s)$ be a unit speed curve and T be its unit tangent vector. In this case, the geodesic curvature of $\alpha(s)$ is defined, [10], as

$$\rho_g = \|D_T T\| = \left\|\frac{d^2\alpha}{ds^2}\right\|. \tag{12}$$

3. The characterizations between Darboux and the Bishop frames of type-2 in E_1^3

Theorem 3.1. *Let $\{T, g, N\}$ and $\{\Omega_1, \Omega_2, B\}$ be Darboux and the Bishop frames of type-2, respectively. There exists a relation between them as*

$$\begin{bmatrix} T \\ g \\ N \end{bmatrix} = \begin{bmatrix} \sinh\overline{\theta} & \cosh\overline{\theta} & 0 \\ \cosh\theta\cosh\overline{\theta} & \cosh\theta\sinh\overline{\theta} & \sinh\theta \\ \sinh\theta\cosh\overline{\theta} & \sinh\theta\sinh\overline{\theta} & \cosh\theta \end{bmatrix} \begin{bmatrix} \Omega_1 \\ \Omega_2 \\ B \end{bmatrix}, \tag{13}$$

where θ is the angle between the vectors g and n, and $\overline{\theta} = \arctan h\dfrac{\xi_2}{\xi_1}$.

Proof. Substituting (8) into (3) gives the matrix (13). □

Theorem 3.2. *Let $\{T, n, B\}$ and $\{\Omega_1, \Omega_2, B\}$ be Frenet and the Bishop frames of type-2 and $\widehat{w}$, $\overline{w}$ be instantaneous vectors of thee trihedrons, respectively. There is a relation between these instantaneous vectors as follows*

$$\overline{w} = \widehat{w} - \rho B. \tag{14}$$

Proof. Let $\widehat{w}$ and $\overline{w}$ be instantaneous vectors of Frenet and Bishop trihedrons of type-2. By the definition of Darboux vector in sense of Frenet frame and the equation (10), we may write

$$\begin{cases} B' = \widehat{w} \wedge B, \\ B' = \overline{w} \wedge B. \end{cases} \tag{15}$$

By (15), we write

$$(\widehat{w} - \overline{w}) \wedge B = 0. \tag{16}$$

By virtue of (16), we have

$$\overline{w} = \widehat{w} - \lambda B. \tag{17}$$

Substituting (11) and (14) into (17) gives

$$\lambda = \rho,$$

which completes the proof. □

Corollary 3.1. *If $\rho = 0$, then $\overline{w} = \widehat{w}$.*

Proof. Substituting $\rho = 0$ into (17) gives the result. □

Theorem 3.3. *Let $\varphi(u, v)$ be a smooth timelike surface in E_1^3 and $\alpha(s)$ be a regular spacelike curve on it. The type-2 Bishop curvatures of the curve $\alpha(s)$ are expressed in terms of invariants of the Darboux frame as*

$$\begin{aligned} \xi_1 &= \rho_g \sinh\theta \sinh\overline{\theta} - \rho_n \cosh\theta \sinh\overline{\theta} + \theta' \cosh\overline{\theta}, \\ \xi_2 &= \rho_g \cosh\overline{\theta} \sinh\theta - \rho_n \cosh\overline{\theta} \cosh\theta + \frac{\theta' \cosh^2\overline{\theta} \sinh\overline{\theta}}{1 - \sinh^2\overline{\theta}}. \end{aligned} \tag{18}$$

Proof. By Theorem 4.1, we have

$$\begin{aligned} g &= \cosh\theta \cosh\overline{\theta}\Omega_1 + \cosh\theta \sinh\overline{\theta}\Omega_2 + \sinh\theta B, \\ N &= \sinh\theta \cosh\overline{\theta}\Omega_1 + \sinh\theta \sinh\overline{\theta}\Omega_2 + \cosh\theta B. \end{aligned} \tag{19}$$

Differentiating the vector g in (19) gives us

$$\begin{aligned} &(\rho_g \sinh\overline{\theta} + \tau_g \sinh\theta \cosh\overline{\theta})\Omega_1 + (\tau_g \sinh\theta \sinh\overline{\theta} + \rho_g \cosh\overline{\theta})\Omega_2 \\ &+\tau_g \cosh\theta B = (\theta' \sinh\theta \cosh\overline{\theta} + \overline{\theta}' \cosh\theta \sinh\overline{\theta} - \xi_1 \sinh\theta)\Omega_1 \\ &+(\overline{\theta}' \cosh\overline{\theta} \cosh\theta + \theta' \sinh\theta \sinh\overline{\theta} - \xi_2 \sinh\theta)\Omega_2 \\ &+(\xi_1 \cosh\theta \cosh\overline{\theta} - \xi_2 \cosh\theta \sinh\overline{\theta} + \theta' \cosh\theta)B \end{aligned} \tag{20}$$

Using (2) and (7) in (19), we obtain

$$\begin{aligned} \rho_g \sinh\overline{\theta} + \tau_g \sinh\theta \cosh\overline{\theta} &= \theta' \sinh\theta \cosh\overline{\theta} + \overline{\theta}' \cosh\theta \sinh\overline{\theta} - \xi_1 \sinh\theta, \\ \tau_g \sinh\theta \sinh\overline{\theta} + \rho_g \cosh\overline{\theta} &= \overline{\theta}' \cosh\overline{\theta} \cosh\theta + \theta' \sinh\theta \sinh\overline{\theta} - \xi_2 \sinh\theta, \\ \tau_g \cosh\theta &= \xi_1 \cosh\theta \cosh\overline{\theta} - \xi_2 \cosh\theta \sinh\overline{\theta} + \theta' \cosh\theta. \end{aligned} \tag{21}$$

and the torsion of curve is obtained as

$$\tau_g = \xi_1 \cosh\overline{\theta} - \xi_2 \sinh\overline{\theta} + \theta'. \tag{22}$$

Also differentiating the vector N in (19) gives us

$$\begin{aligned} &(\rho_n \sinh\overline{\theta} + \tau_g \cosh\theta \cosh\overline{\theta})\Omega_1 + (\tau_g \cosh\theta \sinh\overline{\theta} + \rho_n \cosh\overline{\theta})\Omega_2 \\ &+\tau_g \sinh\theta B = (\theta' \cosh\theta \cosh\overline{\theta} + \overline{\theta}' \sinh\theta \sinh\overline{\theta} - \xi_1 \cosh\theta)\Omega_1 \\ &+(\overline{\theta}' \cosh\overline{\theta} \sinh\theta + \theta' \cosh\theta \sinh\overline{\theta} - \xi_2 \cosh\theta)\Omega_2 \\ &+(\xi_1 \sinh\theta \cosh\overline{\theta} - \xi_2 \sinh\theta \sinh\overline{\theta} + \theta' \sinh\theta)B \end{aligned} \tag{23}$$

Using (2) and (7) in (23), we obtain

$$\begin{aligned}\rho_n \sinh\overline{\theta} + \tau_g \cosh\theta\cosh\overline{\theta} &= \theta' \cosh\theta\cosh\overline{\theta} + \overline{\theta}' \sinh\theta\sinh\overline{\theta} - \xi_1 \cosh\theta,\\ \tau_g \cosh\theta\sinh\overline{\theta} + \rho_n \cosh\overline{\theta} &= \overline{\theta}' \cosh\overline{\theta}\sinh\theta + \theta' \cosh\theta\sinh\overline{\theta} - \xi_2 \cosh\theta,\\ \tau_g \sinh\theta &= \xi_1 \sinh\theta\cosh\overline{\theta} - \xi_2 \sinh\theta\sinh\overline{\theta} + \theta' \sinh\theta.\end{aligned} \tag{24}$$

and again the torsion of curve is again obtained as similar to (22) as

$$\tau_g = \xi_1 \cosh\overline{\theta} - \xi_2 \sinh\overline{\theta} + \theta'. \tag{25}$$

By (21) and (24), the following values of the type-2 Bishop curvatures ξ_1 and ξ_2 are found as

$$\begin{aligned}\xi_1 &= \theta' \cosh\overline{\theta} + \frac{\overline{\theta}' \cosh\theta\sinh\overline{\theta}}{\sinh\theta} - \rho_g \frac{\sinh\overline{\theta}}{\sinh\theta} - \tau_g \cosh\overline{\theta},\\ \xi_1 &= \theta' \cosh\overline{\theta} + \frac{\overline{\theta}' \sinh\theta\sinh\overline{\theta}}{\cosh\theta} - \rho_n \frac{\sinh\overline{\theta}}{\cosh\theta} - \tau_g \cosh\overline{\theta},\end{aligned} \tag{26}$$

and

$$\begin{aligned}\xi_2 &= \frac{\overline{\theta}' \cosh\overline{\theta}\cosh\theta}{\sinh\theta} + \theta' \sinh\overline{\theta} - \tau_g \sinh\overline{\theta} - \rho_g \frac{\cosh\overline{\theta}}{\sinh\theta},\\ \xi_2 &= \frac{\overline{\theta}' \cosh\overline{\theta}\sinh\theta}{\cosh\theta} + \theta' \sinh\overline{\theta} - \tau_g \sinh\overline{\theta} - \rho_n \frac{\cosh\overline{\theta}}{\cosh\theta}.\end{aligned} \tag{27}$$

The common solution of (26) is

$$\xi_1 = \rho_g \sinh\theta\sinh\overline{\theta} - \rho_n \cosh\theta\sinh\overline{\theta} + \theta' \cosh\overline{\theta}. \tag{28}$$

The common solution of (27) is

$$\xi_2 = \rho_g \cosh\overline{\theta}\sinh\theta - \rho_n \cosh\overline{\theta}\cosh\theta + \frac{\theta' \cosh^2\overline{\theta}\sinh\overline{\theta}}{1 - \sinh^2\overline{\theta}}. \tag{29}$$

□

Corollary 3.2. *Among the curvature functions of the type-2 Bishop frame and Darboux frame, there is the following relation*

$$\xi_1^2 + \xi_2^2 = (\rho_g \sinh\theta - \rho_n \cosh\theta)^2 (\sinh^2\overline{\theta} + \cosh^2\overline{\theta}) + \Theta,$$

where

$$\Theta = \frac{4\theta' \sinh\overline{\theta}\cosh\overline{\theta}(\sinh\theta - \cosh\theta)}{1 - \sinh^2\overline{\theta}} + \frac{\theta'^2 \cosh^2\overline{\theta}(\cosh^2\overline{\theta} + \sinh^2\overline{\theta})}{(1 - \sinh^2\overline{\theta})^2}.$$

Proof. The proof is straightforwardly seen by using (28) and (29). □

Corollary 3.3. *The Bishop Darboux vector of type-2 is expressed in terms of Darboux invariants as follows*

$$\overline{w} = (\rho_g \cosh\overline{\theta}\sinh\theta - \rho_n \cosh\overline{\theta}\cosh\theta + \frac{\theta' \cosh^2\overline{\theta}\sinh\overline{\theta}}{1 - \sinh^2\overline{\theta}})\Omega_1. \tag{30}$$

Proof. Replacing (18) in (14) gives the result (30). □

Corollary 3.4. ***(i)*** *If the curve $\alpha(s)$ is asymptotic on the surface $\varphi(u, v)$, then $\rho_n = 0$. Therefore the formulas (18) turns into the following form*

$$\begin{aligned}\xi_1 &= \rho_g \sinh\theta\sinh\overline{\theta} + \theta' \cosh\overline{\theta},\\ \xi_2 &= \rho_g \cosh\overline{\theta}\sinh\theta + \frac{\theta' \cosh^2\overline{\theta}\sinh\overline{\theta}}{1 - \sinh^2\overline{\theta}}.\end{aligned} \tag{31}$$

(ii) *If the curve $\alpha(s)$ is geodesic on the surface $\varphi(u,v)$, then $\rho_g = 0$. Therefore the formulas (18) turns into the following form*

$$\begin{aligned} \xi_1 &= -\rho_n \cosh\theta \sinh\overline{\theta} + \theta' \cosh\overline{\theta}, \\ \xi_2 &= -\rho_n \cosh\overline{\theta} \cosh\theta + \frac{\theta' \cosh^2\overline{\theta} \sinh\overline{\theta}}{1-\sinh^2\overline{\theta}}. \end{aligned} \tag{32}$$

Proof. The proof is seen by using $\rho_n = 0$ and $\rho_g = 0$ in (18). □

Theorem 3.4. *Let $\{T, g, N\}$ and $\{\Omega_1, \Omega_2, B\}$ and be Darboux and the Bishop frames of type-2, respectively. The geodesic torsion of the curve $\alpha(s)$ is given as*

$$\tau_g = \frac{\rho_n \sinh\overline{\theta} \sinh^2\theta - \rho_g \sinh\theta \sinh\overline{\theta} \cosh\theta}{\cosh\theta \cosh\overline{\theta}}. \tag{33}$$

Proof. By Theorem 4.1, we write that

$$N.\Omega_1 = \sinh\theta \cosh\overline{\theta}. \tag{34}$$

Differentiating (34) gives

$$\rho_n T\Omega_1 + \tau_g g\Omega_1 + \xi_1 \cosh\theta = \theta' \cosh\theta \cosh\overline{\theta} + \overline{\theta}' \sinh\theta \sinh\overline{\theta}, \tag{35}$$

then after some calculations, we obtain (33). □

3.1. **Spherical image of Darboux vector.**

Theorem 3.5. *Let $\alpha(s)$ be a spacelike curve on the surface $\varphi(u,v)$ in E_1^3 and and $\{\Omega_1, \Omega_2, B\}$ be the type-2 Bishop frame. Let $\overline{w}$ be a unit Darboux vector of the curve α with respect to the type-2 Bishop frame. In this case, the geodesic curvature $\rho_{\overline{w}}^{II}$ of the spherical image of $\overline{w}$ is found as follows*

$$\rho_{\overline{w}}^{II} = \sqrt{\Delta^2 - \Psi^2 + \Pi^2}, \tag{36}$$

where

$$\begin{aligned} \Delta = [&-\frac{\Gamma'}{\Gamma^3}(\tau' \sinh\widetilde{\theta} + \tau\widetilde{\theta}' \cosh\widetilde{\theta}) + \frac{1}{\Gamma^2}(\tau'' \sinh\widetilde{\theta} + 2\tau'\widetilde{\theta}' \cosh\widetilde{\theta} \\ &+\tau\widetilde{\theta}'' \cosh\widetilde{\theta} + \tau\widetilde{\theta}'^2 \sinh\widetilde{\theta} - \tau^2 \sinh\widetilde{\theta} \cosh\widetilde{\theta}\xi_1)], \end{aligned}$$

$$\Psi = [-\frac{1}{\Gamma^2}\tau^2 \sinh\widetilde{\theta} \cosh\widetilde{\theta}\xi_2],$$

$$\begin{aligned} \Pi = [&-\frac{\Gamma'}{\Gamma^3}\tau^2 \sinh\widetilde{\theta} \cosh\widetilde{\theta} + \frac{1}{\Gamma^2}(\tau' \sinh\widetilde{\theta} + \tau\widetilde{\theta}' \cosh\widetilde{\theta} \\ &+2\tau\tau' \sinh\widetilde{\theta} \cosh\widetilde{\theta} + \tau^2\widetilde{\theta}'(\sinh^2\widetilde{\theta} + \cosh^2\widetilde{\theta}))], \end{aligned}$$

and

$$\Gamma = (\tau' \sinh\widetilde{\theta} + \tau\widetilde{\theta}' \cosh\widetilde{\theta})^2 + \tau^4 \sinh^2\widetilde{\theta} \cosh^2\widetilde{\theta},$$

and the angle $\widetilde{\theta}$ belongs to the spherical curve $\overline{w} = \beta(s_{\overline{w}})$.

Proof. Let $\beta = \beta(s_{\overline{w}})$ be the spherical image curve of the Darboux vector of α according to the type-2 Bishop frame. One can write that

$$\overline{w}(s) = E_1(s_1) = \xi_2\Omega_1 = \tau \sinh\widetilde{\theta}\Omega_1,$$

where

$$\sinh\widetilde{\theta} = \frac{\xi_2}{\sqrt{|\xi_1^2 - \xi_2^2|}}.$$

Thus we can also write

$$\beta(s_{\overline{w}}) = \tau \sinh \widetilde{\theta} \Omega_1, \tag{37}$$

then by differentiating (37) with respect to $s_{\overline{w}}$, we have

$$\frac{d\beta}{ds_{\overline{w}}} = ((\tau' \sinh \widetilde{\theta} + \tau \widetilde{\theta}' \cosh \widetilde{\theta})\Omega_1 + \tau^2 \sinh \widetilde{\theta} \cosh \widetilde{\theta} B)\frac{ds}{ds_{\overline{w}}}, \tag{38}$$

that is, the tangent vector T of spherical curve $\beta(s_{\overline{w}})$ is as in (38). By taking the norm of (38), we obtain

$$\frac{ds}{ds_{\overline{w}}} = \frac{1}{\Gamma}, \tag{39}$$

where

$$\Gamma = (\tau' \sinh \widetilde{\theta} + \tau \widetilde{\theta}' \cosh \widetilde{\theta})^2 + \tau^4 \sinh^2 \widetilde{\theta} \cosh^2 \widetilde{\theta}.$$

Using (39) in (38), we obtain

$$T_{\overline{w}} = \frac{1}{\Gamma}[(\tau' \sinh \widetilde{\theta} + \tau \widetilde{\theta}' \cosh \widetilde{\theta})\Omega_1 + \tau^2 \sinh \widetilde{\theta} \cosh \widetilde{\theta} B]. \tag{40}$$

Differentiating (40) gives

$$\begin{aligned}\frac{dT}{ds_{\overline{w}}} = {} & [-\frac{\Gamma'}{\Gamma^3}(\tau' \sinh \widetilde{\theta} + \tau \widetilde{\theta}' \cosh \widetilde{\theta}) + \frac{1}{\Gamma^2}(\tau'' \sinh \widetilde{\theta} + 2\tau' \widetilde{\theta}' \cosh \widetilde{\theta} \\ & + \tau \widetilde{\theta}'' \cosh \widetilde{\theta} + \tau \widetilde{\theta}'^2 \sinh \widetilde{\theta} - \tau^2 \sinh \widetilde{\theta} \cosh \widetilde{\theta} \xi_1)]\Omega_1 \\ & + [-\frac{1}{\Gamma^2}\tau^2 \sinh \widetilde{\theta} \cosh \widetilde{\theta} \xi_2]\Omega_2 + [-\frac{\Gamma'}{\Gamma^3}\tau^2 \sinh \widetilde{\theta} \cosh \widetilde{\theta} \\ & + \frac{1}{\Gamma^2}(\tau' \sinh \widetilde{\theta} + \tau \widetilde{\theta}' \cosh \widetilde{\theta} + 2\tau\tau' \sinh \widetilde{\theta} \cosh \widetilde{\theta} \\ & + \tau^2 \widetilde{\theta}'(\sinh^2 \widetilde{\theta} + \cosh^2 \widetilde{\theta}))]B.\end{aligned} \tag{41}$$

By taking the norm of (41), we find the result (36). □

Also let's express Frenet-Serret apparatus of the Darboux spherical image. We know that the equation (40) is the tangent vector and the first curvature is equal to (36). Using (41), we have the principal normal vector

$$\begin{aligned}n_{\overline{w}} = {} & \frac{1}{\rho_{\overline{w}}}\{[-\frac{\Gamma'}{\Gamma^3}(\tau' \sinh \widetilde{\theta} + \tau \widetilde{\theta}' \cosh \widetilde{\theta}) + \frac{1}{\Gamma^2}(\tau'' \sinh \widetilde{\theta} + 2\tau' \widetilde{\theta}' \cosh \widetilde{\theta} \\ & + \tau \widetilde{\theta}'' \cosh \widetilde{\theta} + \tau \widetilde{\theta}'^2 \sinh \widetilde{\theta} - \tau^2 \sinh \widetilde{\theta} \cosh \widetilde{\theta} \xi_1)]\Omega_1 \\ & + [-\frac{1}{\Gamma^2}\tau^2 \sinh \widetilde{\theta} \cosh \widetilde{\theta} \xi_2]\Omega_2 + [-\frac{\Gamma'}{\Gamma^3}\tau^2 \sinh \widetilde{\theta} \cosh \widetilde{\theta} \\ & + \frac{1}{\Gamma^2}(\tau' \sinh \widetilde{\theta} + \tau \widetilde{\theta}' \cosh \widetilde{\theta} + 2\tau\tau' \sinh \widetilde{\theta} \cosh \widetilde{\theta} \\ & + \tau^2 \widetilde{\theta}'(\sinh^2 \widetilde{\theta} + \cosh^2 \widetilde{\theta}))]B\}.\end{aligned} \tag{42}$$

The product $T_{\overline{w}} \wedge n_{\overline{w}}$ gives us the binormal vector according to the type-2 Bishop frame as follows:

$$\begin{aligned}T_{\overline{w}} \times n_{\overline{w}} = {} & -\frac{1}{\rho_{\overline{w}}\Gamma^3}\tau^4 \sinh^2 \widetilde{\theta} \cosh^2 \widetilde{\theta} \xi_2 \Omega_1 + \frac{1}{\rho_{\overline{w}}\Gamma^3}(\tau^2 \tau'' \sinh^2 \widetilde{\theta} \cosh \widetilde{\theta} \\ & + \tau^3 \widetilde{\theta}'' \sinh \widetilde{\theta} \cosh^2 \widetilde{\theta} - \tau^4 \sinh^2 \widetilde{\theta} \cosh^2 \widetilde{\theta} \xi_1 - \tau'^2 \sinh^2 \widetilde{\theta} \\ & - 2\tau\tau' \widetilde{\theta}' \sinh \widetilde{\theta} \cosh \widetilde{\theta} - 2\tau\tau'^2 \sinh^2 \widetilde{\theta} \cosh \widetilde{\theta} - \tau^2 \tau' \widetilde{\theta}' \sinh^3 \widetilde{\theta} \\ & - \tau^2 \tau' \widetilde{\theta}' \sinh \widetilde{\theta} \cosh^2 \widetilde{\theta} - \tau^2 \widetilde{\theta}'^2 \cosh^2 \widetilde{\theta} - \tau^3 \widetilde{\theta}'^2 \cosh^3 \widetilde{\theta})\Omega_2 \\ & + \frac{1}{\rho_{\overline{w}}\Gamma^3}\tau^2 \sinh \widetilde{\theta} \cosh \widetilde{\theta} \xi_2 (\tau' \sinh \widetilde{\theta} + \tau \widetilde{\theta}' \cosh \widetilde{\theta}) B\end{aligned} \tag{43}$$

Let's calculate the torsion of Darboux spherical image $\tau_{\overline{w}}$. But first let's denote (43) as

$$b_{\overline{w}} = \overline{\lambda}\Omega_1 + \overline{\gamma}\Omega_2 + \overline{\psi}B, \tag{44}$$

where

$$\overline{\lambda} = -\frac{1}{\rho_{\overline{w}}\Gamma^3}\tau^4 \sinh^2\widetilde{\theta}\cosh^2\widetilde{\theta}\xi_2,$$

$$\begin{aligned}\overline{\gamma} = \frac{1}{\rho_{\overline{w}}\Gamma^3}&(\tau^2\tau''\sinh^2\widetilde{\theta}\cosh\widetilde{\theta} + \tau^3\widetilde{\theta}''\sinh\widetilde{\theta}\cosh^2\widetilde{\theta}\\ &-\tau^4\sinh^2\widetilde{\theta}\cosh^2\widetilde{\theta}\xi_1 - \tau'^2\sinh^2\widetilde{\theta} - 2\tau\tau'\widetilde{\theta}'\sinh\widetilde{\theta}\cosh\widetilde{\theta}\\ &-2\tau\tau'^2\sinh^2\widetilde{\theta}\cosh\widetilde{\theta} - \tau^2\tau'\widetilde{\theta}'\sinh^3\widetilde{\theta} - \tau^2\tau'\widetilde{\theta}'\sinh\widetilde{\theta}\cosh^2\widetilde{\theta}\\ &-\tau^2\widetilde{\theta}'^2\cosh^2\widetilde{\theta} - \tau^3\widetilde{\theta}'^2\cosh^3\widetilde{\theta}),\end{aligned}$$

$$\overline{\psi} = \frac{1}{\rho_{\overline{w}}\Gamma^3}\tau^2\sinh\widetilde{\theta}\cosh\widetilde{\theta}\xi_2(\tau'\sinh\widetilde{\theta} + \tau\widetilde{\theta}'\cosh\widetilde{\theta}).$$

Differentiating (44) gives

$$\frac{db_{\overline{w}}}{ds_{\overline{w}}} = \frac{(\overline{\lambda}' - \overline{\psi}\xi_1)}{\Gamma}\Omega_1 + \frac{(\overline{\gamma}' - \overline{\psi}\xi_2)}{\Gamma}\Omega_2 + \frac{(\overline{\lambda}\xi_1 - \overline{\gamma}\xi_2 + \overline{\psi}')}{\Gamma}B, \tag{45}$$

by using (42) and (45), we have the torsion of Darboux spherical image $\tau_{\overline{w}}$ as follows:

$$\begin{aligned}\tau_{\overline{w}} = \{&-\frac{\Gamma'}{\rho_{\overline{w}}\Gamma^4}(\tau'\sinh\widetilde{\theta} + \tau\widetilde{\theta}'\cosh\widetilde{\theta}) + \frac{1}{\rho_{\overline{w}}\Gamma^3}(\tau''\sinh\widetilde{\theta} + 2\tau'\widetilde{\theta}'\cosh\widetilde{\theta}\\ &+\tau\widetilde{\theta}''\cosh\widetilde{\theta} + \tau\widetilde{\theta}'^2\sinh\widetilde{\theta} - \tau^2\sinh\widetilde{\theta}\cosh\widetilde{\theta}\xi_1)\}(\overline{\lambda}' - \overline{\psi}\xi_1)\\ &+\frac{1}{\rho_{\overline{w}}\Gamma^3}\tau^2\sinh\widetilde{\theta}\cosh\widetilde{\theta}\xi_2(\overline{\gamma}' - \overline{\psi}\xi_2) + \{-\frac{\Gamma'}{\Gamma^3}\tau^2\sinh\widetilde{\theta}\cosh\widetilde{\theta}\\ &+\frac{1}{\Gamma^3}(\tau'\sinh\widetilde{\theta} + \tau\widetilde{\theta}'\cosh\widetilde{\theta} + 2\tau\tau'\sinh\widetilde{\theta}\cosh\widetilde{\theta}\\ &+\tau^2\widetilde{\theta}'(\sinh^2\widetilde{\theta} + \cosh^2\widetilde{\theta}))\}(\overline{\lambda}\xi_1 - \overline{\gamma}\xi_2 + \psi')\end{aligned} \tag{46}$$

Example 3.1. *Let us consider the following surface*

$$\varphi(s, v) = (\sinh s, \cosh s, v). \tag{47}$$

By taking $s = v$ in (47), we have the curve α on the surface φ as follows:

$$\alpha(s) = (\sinh s, \cosh s, s). \tag{48}$$

The parameterizations of curve and surface in (47) and (48) are plotted together as in Figure 1:

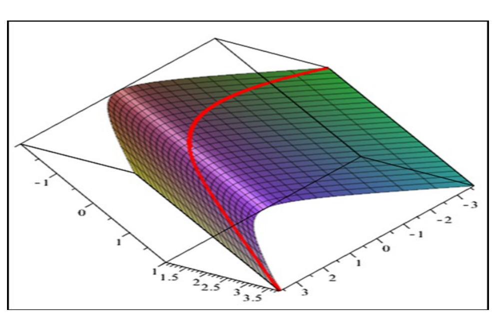

FIGURE 1.

The Frenet-Serret frame vectors of the curve $\alpha(s)$ are found as follows:

$$\begin{aligned} T &= (\cosh s, \sinh s, 1), \\ n &= (\sinh s, \cosh s, 0), \\ B &= (-\cosh s, \sinh s, 1). \end{aligned} \tag{49}$$

The first and second curvatures of the curve $\alpha(s)$ are obtained as follows:

$$\rho = 1, \quad \tau = -1. \tag{50}$$

We form the finite integral which is a key of the Bishop trihedra of type-2 as follows:

$$\overline{\theta}(s) = \int_0^s ds = s. \tag{51}$$

The type-2 Bishop curvatures of the curve $\alpha(s)$ are found as

$$\xi_1 = -\cosh s, \quad \xi_2 = -\sinh s.$$

Using (50) and (51) in (8), we obtain the Bishop frame vectors Ω_1, Ω_2 as follows:

$$\Omega_1 = (0, 1, -\sinh s), \tag{52}$$

and

$$\Omega_2 = (-1, 0, -\cosh s) \tag{53}$$

By (52) and (53), the equation of the spherical image $\beta(s_{\overline{w}})$ is found as follows:

$$\beta(s_{\overline{w}}) = (0, -\sinh s, \sinh^2 s). \tag{54}$$

The Bishop spherical curve of the curve (48) is plotted together as in Figure 4:

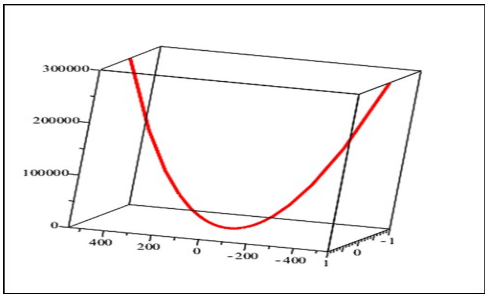

FIGURE 2. Bishop spherical image of the curve α on S_0^2

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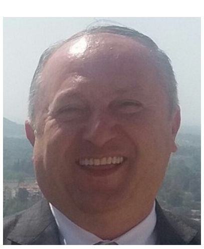

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