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Extended q -Dedekind-type Daehee-Changhee sums associated with extended q -Euler polynomials

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Abstract

In the present paper, we aim to specify a p -adic continuous function for an odd prime inside a p -adic q -analog of the extended Dedekind-type sums of higher order according to extended q -Euler polynomials (or weighted q -Euler polynomials) which is derived from a fermionic p -adic q -deformed integral on \mathbb{Z}_p .

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1 Introduction

Let p be chosen as a fixed odd prime number. In this paper \mathbb{Z}_p , \mathbb{Q}_p , \mathbb{C} and \mathbb{C}_p will, respectively, denote the ring of p -adic rational integers, the field of p -adic rational numbers, the complex numbers, and the completion of an algebraic closure of \mathbb{Q}_p .

Let v_p be a normalized exponential valuation of \mathbb{C}_p by

$$|p|_p = p^{-v_p(p)} = \frac{1}{p}.$$

When one talks of a q -extension, q is variously considered as an indeterminate, a complex number $q \in \mathbb{C}$ or a p -adic number $q \in \mathbb{C}_p$. If $q \in \mathbb{C}$, we assume that $|q| < 1$. If $q \in \mathbb{C}_p$, we assume that $|1 - q|_p < 1$ (see, for details, [1–16]).

The following measure is defined by Kim: for any positive integer n and $0 \leq a < p^n$,

$$\mu_q(a + p^n\mathbb{Z}_p) = (-q)^a \frac{(1+q)}{1+q^{p^n}},$$

which can be extended to a measure on \mathbb{Z}_p (for details, see [5–11]).

Extended q -Euler polynomials (also known as weighted q -Euler polynomials) are defined by

$$\tilde{E}_{n,q}^{(\alpha)}(x) = \int_{\mathbb{Z}_p} \left(\frac{1 - q^{\alpha(x+\xi)}}{1 - q^\alpha} \right)^n d\mu_q(\xi) \tag{1}$$

for $n \in \mathbb{Z}_+ := \{0, 1, 2, 3, \dots\}$. We note that

$$\lim_{q \rightarrow 1} \tilde{E}_{n,q}^{(\alpha)}(x) = E_n(x),$$

where $E_n(x)$ are n th Euler polynomials, which are defined by the rule

$$\sum_{n=0}^{\infty} E_n(x) \frac{t^n}{n!} = e^{tx} \frac{2}{e^t + 1}, \quad |t| < \pi$$

(for details, see [13]). In the case $x = 0$ in (1), then we have $\tilde{E}_{n,q}^{(\alpha)}(0) := \tilde{E}_{n,q}^{(\alpha)}$, which are called extended q -Euler numbers (or weighted q -Euler numbers).

Extended q -Euler numbers and polynomials have the following explicit formulas:

$$\tilde{E}_{n,q}^{(\alpha)} = \frac{1+q}{(1-q^\alpha)^n} \sum_{l=0}^n \binom{n}{l} (-1)^l \frac{1}{1+q^{\alpha l+1}}, \tag{2}$$

$$\tilde{E}_{n,q}^{(\alpha)}(x) = \frac{1+q}{(1-q^\alpha)^n} \sum_{l=0}^n \binom{n}{l} (-1)^l \frac{q^{\alpha l x}}{1+q^{\alpha l+1}}, \tag{3}$$

$$\tilde{E}_{n,q}^{(\alpha)}(x) = \sum_{l=0}^n \binom{n}{l} q^{\alpha l x} \tilde{E}_{l,q}^{(\alpha)} \left(\frac{1-q^{\alpha x}}{1-q^\alpha} \right)^{n-l}. \tag{4}$$

Moreover, for $d \in \mathbb{N}$ with $d \equiv 1 \pmod{2}$,

$$\tilde{E}_{n,q}^{(\alpha)}(x) = \left(\frac{1+q}{1+q^d} \right) \left(\frac{1-q^{\alpha d}}{1-q^\alpha} \right)^n \sum_{a=0}^{d-1} (-1)^a \tilde{E}_{n,q}^{(\alpha)} \left(\frac{x+a}{d} \right); \tag{5}$$

see [13].

For any positive integer h, k and m , Dedekind-type DC sums are given by Kim in [5, 6], and [7] as follows:

$$S_m(h, k) = \sum_{M=1}^{k-1} (-1)^{M-1} \frac{M}{k} \bar{E}_m \left(\frac{hM}{k} \right),$$

where $\bar{E}_m(x)$ are m th periodic Euler functions.

Kim [6] derived some interesting properties for Dedekind-type DC sums and considered a p -adic continuous function for an odd prime number to contain a p -adic q -analog of the higher order Dedekind-type DC sums $k^m S_{m+1}(h, k)$. Simsek [15] gave a q -analog of Dedekind-type sums and derived interesting properties. Furthermore, Araci *et al.* studied Dedekind-type sums in accordance with modified q -Euler polynomials with weight α [14], modified q -Genocchi polynomials with weight α [4], and weighted q -Genocchi polynomials [16].

Recently, weighted q -Bernoulli numbers and polynomials were first defined by Kim in [11]. Next, many mathematicians, by utilizing Kim's paper [11], have introduced various generalization of some known special polynomials such as Bernoulli polynomials, Euler polynomials, Genocchi polynomials, and so on, which are called weighted q -Bernoulli, weighted q -Euler, and weighted q -Genocchi polynomials in [1, 2, 11–13].

By the same motivation of the above knowledge, we give a weighted p -adic q -analog of the higher order Dedekind-type DC sums $k^m S_{m+1}(h, k)$ which are derived from a fermionic p -adic q -deformed integral on \mathbb{Z}_p .

2 Extended q -Dedekind-type sums associated with extended q -Euler polynomials

Let w be the Teichmüller character (mod p). For $x \in \mathbb{Z}_p^* := \mathbb{Z}_p \setminus p\mathbb{Z}_p$, set

$$\langle x : q \rangle = w^{-1}(x) \left(\frac{1 - q^x}{1 - q} \right).$$

Let a and N be positive integers with $(p, a) = 1$ and $p \mid N$. We now consider

$$\tilde{C}_q^{(\alpha)}(s, a, N : q^N) = w^{-1}(a) (a : q^\alpha)^s \sum_{j=0}^{\infty} \binom{s}{j} q^{\alpha a j} \left(\frac{1 - q^{\alpha N}}{1 - q^{\alpha a}} \right)^j \tilde{E}_{j, q^N}^{(\alpha)}.$$

In particular, if $m + 1 \equiv 0 \pmod{p - 1}$, then

$$\begin{aligned} \tilde{C}_q^{(\alpha)}(m, a, N : q^N) &= \left(\frac{1 - q^{\alpha a}}{1 - q^\alpha} \right)^m \sum_{j=0}^m \binom{m}{j} q^{\alpha a j} \tilde{E}_{j, q^N}^{(\alpha)} \left(\frac{1 - q^{\alpha N}}{1 - q^{\alpha a}} \right)^j \\ &= \left(\frac{1 - q^{\alpha N}}{1 - q^\alpha} \right)^m \int_{\mathbb{Z}_p} \left(\frac{1 - q^{\alpha N(\xi + \frac{a}{N})}}{1 - q^{\alpha N}} \right)^m d\mu_{q^N}(\xi). \end{aligned}$$

Thus, $\tilde{C}_q^{(\alpha)}(m, a, N : q^N)$ is a continuous p -adic extension of

$$\left(\frac{1 - q^{\alpha N}}{1 - q^\alpha} \right)^m \tilde{E}_{m, q^N}^{(\alpha)} \left(\frac{a}{N} \right).$$

Let $[\cdot]$ be the Gauss symbol and let $\{x\} = x - [x]$. Thus, we are now ready to introduce the q -analog of the higher order Dedekind-type DC sums $\tilde{J}_{m, q}^{(\alpha)}(h, k : q^l)$ by the rule

$$\tilde{J}_{m, q}^{(\alpha)}(h, k : q^l) = \sum_{M=1}^{k-1} (-1)^{M-1} \left(\frac{1 - q^{\alpha M}}{1 - q^{\alpha k}} \right) \int_{\mathbb{Z}_p} \left(\frac{1 - q^{\alpha(l\xi + l(\frac{hM}{k})}}{1 - q^{\alpha l}} \right)^m d\mu_{q^l}(\xi).$$

If $m + 1 \equiv 0 \pmod{p - 1}$,

$$\begin{aligned} &\left(\frac{1 - q^{\alpha k}}{1 - q^\alpha} \right)^{m+1} \sum_{M=1}^{k-1} (-1)^{M-1} \left(\frac{1 - q^{\alpha M}}{1 - q^{\alpha k}} \right) \int_{\mathbb{Z}_p} \left(\frac{1 - q^{\alpha k(\xi + \frac{hM}{k})}}{1 - q^{\alpha k}} \right)^m d\mu_{q^k}(\xi) \\ &= \sum_{M=1}^{k-1} (-1)^{M-1} \left(\frac{1 - q^{\alpha M}}{1 - q^\alpha} \right) \left(\frac{1 - q^{\alpha k}}{1 - q^\alpha} \right)^m \int_{\mathbb{Z}_p} \left(\frac{1 - q^{\alpha k(\xi + \frac{hM}{k})}}{1 - q^{\alpha k}} \right)^m d\mu_{q^k}(\xi), \end{aligned}$$

where $p \mid k, (hM, p) = 1$ for each M . By (1), we easily state the following:

$$\begin{aligned} &\left(\frac{1 - q^{\alpha k}}{1 - q^\alpha} \right)^{m+1} \tilde{J}_{m, q}^{(\alpha)}(h, k : q^k) \\ &= \sum_{M=1}^{k-1} \left(\frac{1 - q^{\alpha M}}{1 - q^\alpha} \right) \left(\frac{1 - q^{\alpha k}}{1 - q^\alpha} \right)^m (-1)^{M-1} \end{aligned}$$

$$\begin{aligned} & \times \int_{\mathbb{Z}_p} \left(\frac{1 - q^{\alpha k(\xi + \frac{hM}{k})}}{1 - q^{\alpha k}} \right)^m d\mu_{q^k}(\xi) \\ & = \sum_{M=1}^{k-1} (-1)^{M-1} \left(\frac{1 - q^{\alpha M}}{1 - q^{\alpha}} \right) \tilde{C}_q^{(\alpha)}(m, (hM)_k : q^k), \end{aligned} \tag{6}$$

where $(hM)_k$ denotes the integer x such that $0 \leq x < n$ and $x \equiv \alpha \pmod{k}$.

It is not difficult to indicate the following:

$$\begin{aligned} & \int_{\mathbb{Z}_p} \left(\frac{1 - q^{\alpha(x+\xi)}}{1 - q^{\alpha}} \right)^k d\mu_q(\xi) \\ & = \left(\frac{1 - q^{\alpha m}}{1 - q^{\alpha}} \right)^k \frac{1 + q}{1 + q^m} \sum_{i=0}^{m-1} (-1)^i \int_{\mathbb{Z}_p} \left(\frac{1 - q^{\alpha m(\xi + \frac{x+i}{m})}}{1 - q^{\alpha m}} \right)^k d\mu_{q^m}(\xi). \end{aligned} \tag{7}$$

On account of (6) and (7), we easily see that

$$\begin{aligned} & \left(\frac{1 - q^{\alpha N}}{1 - q^{\alpha}} \right)^m \int_{\mathbb{Z}_p} \left(\frac{1 - q^{\alpha N(\xi + \frac{a}{N})}}{1 - q^{\alpha N}} \right)^m d\mu_{q^N}(\xi) \\ & = \frac{1 + q^N}{1 + q^{Np}} \sum_{i=0}^{p-1} (-1)^i \left(\frac{1 - q^{\alpha Np}}{1 - q^{\alpha}} \right)^m \int_{\mathbb{Z}_p} \left(\frac{1 - q^{\alpha pN(\xi + \frac{a+iN}{pN})}}{1 - q^{\alpha pN}} \right)^m d\mu_{q^{pN}}(\xi). \end{aligned} \tag{8}$$

Because of (6), (7), and (8), we develop the p -adic integration as follows:

$$\tilde{C}_q^{(\alpha)}(s, a, N : q^N) = \frac{1 + q^N}{1 + q^{Np}} \sum_{\substack{0 \leq i \leq p-1 \\ a+iN \not\equiv 0 \pmod{p}}} (-1)^i \tilde{C}_q^{(\alpha)}(s, (a + iN)_{pN}, p^N : q^{pN}).$$

So,

$$\begin{aligned} \tilde{C}_q^{(\alpha)}(m, a, N : q^N) & = \left(\frac{1 - q^{\alpha N}}{1 - q^{\alpha}} \right)^m \int_{\mathbb{Z}_p} \left(\frac{1 - q^{\alpha N(\xi + \frac{a}{N})}}{1 - q^{\alpha N}} \right)^m d\mu_{q^N}(\xi) \\ & \quad - \left(\frac{1 - q^{\alpha Np}}{1 - q^{\alpha}} \right)^m \int_{\mathbb{Z}_p} \left(\frac{1 - q^{\alpha pN(\xi + \frac{a+iN}{pN})}}{1 - q^{\alpha pN}} \right)^m d\mu_{q^{pN}}(\xi), \end{aligned}$$

where $(p^{-1}a)_N$ denotes the integer x with $0 \leq x < N$, $px \equiv a \pmod{N}$ and m is integer with $m + 1 \equiv 0 \pmod{p - 1}$. Therefore, we have

$$\begin{aligned} & \sum_{M=1}^{k-1} (-1)^{M-1} \left(\frac{1 - q^{\alpha M}}{1 - q^{\alpha}} \right) \tilde{C}_q^{(\alpha)}(m, hM, k : q^k) \\ & = \left(\frac{1 - q^{\alpha k}}{1 - q^{\alpha}} \right)^{m+1} \tilde{J}_{m,q}^{(\alpha)}(h, k : q^k) - \left(\frac{1 - q^{\alpha k}}{1 - q^{\alpha}} \right)^{m+1} \\ & \quad \times \left(\frac{1 - q^{\alpha kp}}{1 - q^{\alpha k}} \right) \tilde{J}_{m,q}^{(\alpha)}((p^{-1}h), k : q^{pk}), \end{aligned}$$

where $p \nmid k$ and $p \nmid hm$ for each M . Thus, we give the following definition, which seems interesting for further studying the theory of Dedekind sums.

Definition 1 Let h, k be positive integer with $(h, k) = 1, p \nmid k$. For $s \in \mathbb{Z}_p$, we define a p -adic Dedekind-type DC sums as follows:

$$\tilde{J}_{p,q}^{(\alpha)}(s : h, k : q^k) = \sum_{M=1}^{k-1} (-1)^{M-1} \left(\frac{1 - q^{\alpha M}}{1 - q^\alpha} \right) \tilde{C}_q^{(\alpha)}(m, hM, k : q^k).$$

As a result of the above definition, we state the following theorem.

Theorem 2.1 For $m + 1 \equiv 0 \pmod{p - 1}$ and $(p^{-1}a)_N$ denotes the integer x with $0 \leq x < N$, $px \equiv a \pmod{N}$, then we have

$$\begin{aligned} \tilde{J}_{p,q}^{(\alpha)}(s : h, k : q^k) &= \left(\frac{1 - q^{\alpha k}}{1 - q^\alpha} \right)^{m+1} \tilde{J}_{m,q}^{(\alpha)}(h, k : q^k) \\ &\quad - \left(\frac{1 - q^{\alpha k}}{1 - q^\alpha} \right)^{m+1} \left(\frac{1 - q^{\alpha kp}}{1 - q^{\alpha k}} \right) \tilde{J}_{m,q}^{(\alpha)}((p^{-1}h), k : q^{pk}). \end{aligned}$$

In the special case $\alpha = 1$, our applications in theory of Dedekind sums resemble Kim’s results in [6]. These results seem to be interesting for further studies as in [5, 7] and [15].

Competing interests

The authors declare that they have no competing interests.

Authors’ contributions

All authors contributed equally to this work. All authors read and approved the revised manuscript.

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