ORIGINAL PAPER

A NOTE ON SPRAYS OF ISOTROPIC CURVATURE

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Abstract. *A basic goal of this paper is to calculate, Weyl curvature of R-flat (Ricci-flat) spray of isotropic curvature and a locally projectively R-flat (Ricci-flat) spray, which is a projective invariance. Besides, the equivalents of $\bar{E}$-curvature and H-curvature that are closely related to the mean Berwald curvature have been found for a locally projectively R-flat spray of isotropic curvature.*

Keywords: *spray; R-flat (Ricci-flat); Weyl curvature; isotropic curvature; $\bar{E}$-curvature; H-curvature.*

1. INTRODUCTION

Finsler geometry is more various than Riemannian geometry, since apart from the Riemannian quantities, there are several important non-Riemannian quantities in a Finsler manifold. It is one of the important problems in Finsler geometry is about the geometric meanings of these non-Riemannian invariants. During the past to present, many geometers have studied the non-Riemannian invariants.

In projective geometry, a relationship among geometric structures with common geodesics is determined as point sets. There are some quantities in the projective Finsler geometry. One of the most important of them is Weyl curvature tensor which is play an important role in understanding the projective properties of Finsler metrics. In (1921), a projective invariant for Riemannian metrics had been introduced by Weyl. After then, Weyl's projective invariant to Finsler metrics and sprays had been extended by Douglas [1, 2]. Z. I. Szabo proved that a Finsler space is of scalar curvature if and only if the Weyl curvature tensor vanishes identically [3]. A Finsler space where Weyl curvature tensor vanish is characterized as a locally projectively flat Finsler space [4].

A large part of Finsler geometry is spray geometry. A spray consists of geodesics in a path space [1]. There is knowledge about sprays in [5]. For a Finsler metric $F = F(x, y)$ on a smooth manifold M, the geodesics of $F = F(x, y)$ are characterized by the system of equation

$$\frac{d^2x^i}{dt^2} + 2G^i\left(x, \frac{dx}{dt}\right) = 0, \tag{1}$$

where $G^i = G^i(x, y)$ is called the geodesic spray coefficients and denoted by $G^i = \frac{1}{4}g^{il}\{[F^2]_{x^k y^l}y^k - [F^2]_{x^l}\}$. In standard local coordinate system (x^i, y^i) in TM, the vector field

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$$\boldsymbol{G} = y^i \frac{\partial}{\partial x^i} - 2G^i(y)\frac{\partial}{\partial y^i} \tag{2}$$

is called the spray of F.

In spray geometry, there are several important geometrical quantities such as Ricci curvature, Berwald curvature, the mean Berwald curvature, etc. Akbar-Zadeh determines the quantity $\boldsymbol{H}_y = H_{ij} dx^i \otimes dx^j$ which is obtained as the covariant horizontal differentiation of $\boldsymbol{E}$ along geodesics. Then the following expression is hold [6]:

$$H_{ij} = E_{ij;m} y^m. \tag{3}$$

In [5], Shen has found a new non-Riemannian quantity for Finsler metrics that is closely related to the mean Berwald curvature and called it $\bar{\boldsymbol{E}}$-*curvature*. Recently, Li and Shen have introduced a notion of isotropic curvature for spray [7].

The aim of this paper is to calculate Weyl curvature of R-flat (Ricci-flat) spray of isotropic curvature and a locally projectively R-flat (Ricci-flat) spray, which is a projective invariance. Besides, for a locally projectively R-flat spray of isotropic curvature, the equivalents of $\bar{\boldsymbol{E}}$-curvature and $\boldsymbol{H}$-curvature that are closely related to the mean Berwald curvature are found.

2. MATERIALS AND METHODS

The Riemannian curvature $\boldsymbol{R}_y(v) = R^i_k(y) dx^k \otimes \frac{\partial}{\partial x^i}\Big|_p : T_pM$ is family of linear mappings on the tangent space and the trace of R^i_k is called the Ricci curvature, defined by,

$$R^i_k(y) = 2\frac{\partial G^i}{\partial x^k} - \frac{\partial^2 G^i}{\partial x^j \partial y^k} + 2G^j \frac{\partial^2 G^i}{\partial y^j \partial y^k} - \frac{\partial G^i}{\partial y^j}\frac{\partial G^i}{\partial y^k}, \tag{4}$$

and

$$\boldsymbol{Ric}(y) = (n-1)\boldsymbol{R}_y = R^i_i(y), \tag{5}$$

respectively, where R is called the Ricci scalar [5]. In 1926, Berwald extended the notion of Riemannian curvature tensor in Riemannian geometry to sprays [8]. A spray **G** is said to be of scalar curvature if the Riemannian curvature tensor R^i_k satisfies

$$R^i_k = R\delta^i_k - \tau_k y^i, \tag{6}$$

where $R = R(x,y)$ and $\tau_k = \tau_k(x,y)$ with $\tau_k y^k = R$. Particularly, **G** is said to be isotropic curvature if it is of scalar curvature and $\tau_k = \frac{1}{2} R_{.k}$. where "$.k$" denotes the vertical derivative with respect to y^k.

A spray **G** on a smooth manifold M is said to be $R-flat$, if the Riemannian curvature of **G** vanishes (i.e. $\boldsymbol{R} = 0$). It is said to be $Ricci-flat$, if $\boldsymbol{Ric} = 0$ or $R = 0$.

Proposition 2.1. Let **G** be a spray of isotropic curvature on a smooth manifold M. Then **G** is $R-flat$ if and only if it is $Ricci-flat$ [7].

Consider a spray **G** on a smooth manifold M. At every point, if there is a standard local coordinate system (x^i, y^i) in which $\boldsymbol{G} = y^i \frac{\partial}{\partial x^i} - 2G^i(y)\frac{\partial}{\partial y^i}$ with

$$G^i = Py^i, \tag{7}$$

where $P = P(x,y)$ is a local scalar function on TM, then **G** is called a locally projectively flat spray [7].

Lemma 2.2. Let $\boldsymbol{G} = y^i \frac{\partial}{\partial x^i} - 2G^i(y)\frac{\partial}{\partial y^i}$ be a locally projectively flat spray on an open domain $U \subset R^n$, where $G^i = Py^i$ and $P = P(x,y)$. Then, we have the following expression for the Riemannian curvature [9]

$$R^i_k = \frac{Ric}{n-1}\delta^i_k - \left[3\left(PP_{y^k} - P_{x^k}\right) - \frac{Ric_{.k}}{n-1}\right]y^i \tag{8}$$

Corollary 2.3. Let $\boldsymbol{G} = y^i \frac{\partial}{\partial x^i} - 2G^i(y)\frac{\partial}{\partial y^i}$ be a locally projectively flat spray on an open domain $U \subset R^n$, where $G^i = Py^i$ and $P = P(x,y)$. Then

i) $\boldsymbol{G}$ is $R-flat$ if and only if P is a funk function.

ii) $\boldsymbol{G}$ is $Ricci-flat$ if and only if $\left(PP_{y^k} - P_{x^k}\right)y^k = 0$ [9].

For a vector $y \in T_xM_0$, define $\boldsymbol{W}_y(u) = W^i_k(y)u^k \frac{\partial}{\partial x^i}\Big|_x$ by

$$W^i_k(y) = R^i_k - R\delta^i_k - \frac{1}{n+1}\frac{\partial}{\partial y^m}\{R^m_k - R\delta^m_k\}y^i. \tag{9}$$

The linear transformation $\boldsymbol{W}_y : T_xM \to T_xM$ satisfies $\boldsymbol{W}_y(y) = 0$. We call $\boldsymbol{W} = \{\boldsymbol{W}_y\}_{y\in TM_0}$ as the Weyl curvature, where W^i_k is the Weyl curvature coefficient.

For a vector $y \in T_xM_0$, define $\boldsymbol{B}_y : T_xM \otimes T_xM \otimes T_xM \to T_xM$ and $\boldsymbol{E}_y : T_xM \otimes T_xM \to R$ by

$$\boldsymbol{B}_y(u,v,w) = B^i_{jkl}(y)u^jv^kw^l\frac{\partial}{\partial x^i}\Big|_x, \quad \boldsymbol{E}_y(u,v) = E_{jk}(y)u^jv^k, \tag{10}$$

where

$$B^i_{jkl}(y) = \frac{\partial^3 G^i}{\partial y^j \partial y^k \partial y^l}, \quad E_{jk}(y) = \frac{1}{2}B^m_{jkm}, \tag{11}$$

$u = u^i \frac{\partial}{\partial x^i}\Big|_x$, $v = v^j \frac{\partial}{\partial x^j}\Big|_x$, $w = w^k \frac{\partial}{\partial x^k}\Big|_x$, where the expressions $\boldsymbol{B}$ and $\boldsymbol{E}$ are called the Berwald curvature and the mean Berwald curvature, respectively [5]. Besides, the coefficients of the Berwald curvature and the mean Berwald curvature are obtained as follows:

$$B^i_{jml;k}y^k = \frac{1}{3}\left\{\tau_{k.j.l}y^k\delta^i_m + \tau_{k.j.m}y^k\delta^i_l + \tau_{k.l.m}y^k\delta^i_j + \left(R_{.j.m.l} + \tau_{k.j.m.l}y^k\right)y^i\right\} \tag{12}$$

and

$$E_{jk;l} = \frac{n+1}{6}\tau_{k.j.l}y^k, \tag{13}$$

respectively [5]. The definition of $\overline{\boldsymbol{E}}$ −curvature is given $\overline{\boldsymbol{E}}_y : T_xM \otimes T_xM \otimes T_xM \to R$ by

$$\overline{\boldsymbol{E}}_y(u,v,w) = \bar{E}_{jkl}(y)u^jv^kw^l, \tag{14}$$

where $\overline{\boldsymbol{E}}_{ijk} = E_{ij;k}$ [10].

3. RESULTS

Proposition 3.1. Let **G** be a $R-flat$ spray of isotropic curvature on a smooth manifold M, then the Weyl curvature $\boldsymbol{W}=0$.

Proof: Let **G** be a $R-flat$ spray of isotropic curvature on a smooth manifold M. Substituting the expression (6) into the expression (9), then we obtain coefficient of the Weyl curvature as follows:

$$W_k^i(y)=\left(R\delta_k^i-\frac{1}{2}R_{.k}y^i-R\delta_k^i\right)-\frac{1}{n+1}\frac{\partial}{\partial y^m}\left(R\delta_k^m-\frac{1}{2}R_{.k}y^m-R\delta_k^m\right)y^i. \tag{15}$$

The above expression (15) is rearranged as

$$W_k^i(y)=\frac{1}{2}\left(\frac{1}{n+1}\frac{\partial}{\partial y^m}(R_{.k.m}y^m)-nR_{.k}\right)y^i. \tag{16}$$

Since **G** is a $R-flat$ spray of isotropic curvature, according to the expression (16), we arrive at $\boldsymbol{W}=0$.

Proposition 3.2. Let **G** be a $Ricci-flat$ spray of isotropic curvature on a smooth manifold M, then the Weyl curvature $\boldsymbol{W}=0$.

Proof: Let **G** be a $Ricci-flat$ spray of isotropic curvature on a smooth manifold M. By the expressions (5) and (16), we can write as follows:

$$W_k^i(y)=\frac{1}{2(n^2-1)}\big((Ric_{.k.m}y^m)-nRic_{.k}\big)y^i. \tag{17}$$

Since **G** is a $Ricci-flat$ spray of isotropic curvature, then its Weyl curvature $\boldsymbol{W}=0$.

Proposition 3.3. Let $\boldsymbol{G}=y^i\frac{\partial}{\partial x^i}-2G^i(y)\frac{\partial}{\partial y^i}$ be a locally projectively $R-flat$ spray of isotropic curvature on an open domain $U\subset R^n$, where $G^i=Py^i$ and $P=P(x,y)$. Then its Weyl curvature $\boldsymbol{W}=0$.

Proof: Let **G** be a locally projectively $R-flat$ spray of isotropic curvature. Putting the expression (8) into the expression (9), then we find

$$W_k^i(y)=$$
$$R\delta_k^i-\left[3\left(PP_{y^k}-P_{x^k}\right)-R_{.k}\right]y^i-R\delta_k^i-\frac{1}{n+1}\frac{\partial}{\partial y^m}\left\{\begin{matrix}R\delta_k^m-\left[3\left(PP_{y^k}-P_{x^k}\right)-R_{.k}\right]y^m\\ -R\delta_k^m\end{matrix}\right\}y^i. \tag{18}$$

with a straightforward calculation, we get

$$W_k^i(y)=\frac{1}{n+1}\left(\begin{matrix}\left[3\left(PP_{y^k}-P_{x^k}\right)_{.m}-R_{.k.m}\right]y^m\\ -n\left[3\left(PP_{y^k}-P_{x^k}\right)-R_{.k}\right]\end{matrix}\right)y^i. \tag{19}$$

According to the Corollary 2.3, the expression (19) is equal to zero. The proof is completed.

Proposition 3.4. Let $\boldsymbol{G} = y^i \frac{\partial}{\partial x^i} - 2G^i(y)\frac{\partial}{\partial y^i}$ be a locally projectively $Ricci - flat$ spray of isotropic curvature on an open domain $U \subset R^n$, where $G^i = Py^i$ and $P = P(x, y)$. Then its Weyl curvature $\boldsymbol{W} = 0$.

Proof: Let **G** be a locally projectively $Ricci - flat$ spray of isotropic curvature. Using the expression (5), the above expression (19) gives

$$W_k^i(y) = \frac{1}{n+1}\begin{pmatrix} \left[3\left(PP_{y^k} - P_{x^k}\right)_{.m} - \frac{1}{n-1}Ric_{.k.m}\right]y^m \\ -n\left[3\left(PP_{y^k} - P_{x^k}\right) - \frac{1}{n-1}Ric_{.k}\right] \end{pmatrix} y^i. \tag{20}$$

According to the Corollary 2.3, we arrive at $\boldsymbol{W} = 0$.

Proposition 3.5. Let $\boldsymbol{G} = y^i \frac{\partial}{\partial x^i} - 2G^i(y)\frac{\partial}{\partial y^i}$ be a locally projectively $R - flat$ spray of isotropic curvature on an open domain $U \subset R^n$, where $G^i = Py^i$ and $P = P(x, y)$. Then, we have $\overline{\boldsymbol{E}} = 0$.

Proof: Let **G** be a locally projectively $R - flat$ spray of isotropic curvature. We know that

$$E_{jk;l} = \frac{n+1}{6}\left(\tau_{l.j.k} - R_{.j.k.l}\right). \tag{21}$$

Since $\tau_k = \frac{1}{2}R_{.k}$ for the spray of isotropic curvature, the expression (21) is rewritten as

$$E_{jk;l} = -\frac{n+1}{4}R_{.l.j.k}. \tag{22}$$

From the expression (22), if the locally projectively spray **G** of isotropic curvature is a $R - flat$, then the $\overline{\boldsymbol{E}}$ $-$curvature is equal to zero.

Proposition 3.6. Let $\boldsymbol{G} = y^i \frac{\partial}{\partial x^i} - 2G^i(y)\frac{\partial}{\partial y^i}$ be a locally projectively $R - flat$ spray of isotropic curvature on an open domain $U \subset R^n$, where $G^i = Py^i$ and $P = P(x, y)$. Then, we have

$$\boldsymbol{B} = \quad B_{jkl;m}^i = \quad \frac{8}{(n+1)^2}\left\{E_{jm}E_{kl} + E_{jl}E_{km} + E_{jk}E_{lm}\right\}y^i. \tag{23}$$

Proof: Let **G** be a locally projectively $R - flat$ spray of isotropic curvature. Note that

$$\begin{aligned} B_{jkl;m}^i = \tfrac{1}{3}\{&\left(\tau_{m.j.k} - R_{.j.k.m}\right)\delta_l^i + \left(\tau_{m.j.l} - R_{.j.l.m}\right)\delta_k^i + \left(\tau_{m.k.l} - R_{.k.l.m}\right)\delta_j^i \; + \\ &\left(\tau_{.m.j.k.l} - R_{.j.k.l.m}\right)y^i\} + \tfrac{8}{(n+1)^2}\left\{E_{jm}E_{kl} + E_{jl}E_{km} + E_{jk}E_{lm}\right\}y^i. \end{aligned} \tag{24}$$

Rewritting the expression (24) for the locally projectively $R - flat$ spray of isotropic curvature, then we obtain

$$B^i_{jkl;m} = \frac{1}{3}\Big\{\Big(-\frac{1}{2}R_{.m.j.k} - R_{.j.k.m}\Big)\delta^i_l + \Big(-\frac{1}{2}R_{.m.j.l} - R_{.j.l.m}\Big)\delta^i_k + \Big(-\frac{1}{2}R_{.m.k.l} - R_{.k.l.m}\Big)\delta^i_j + \Big(-\frac{1}{2}R_{.m.j.k.l} - R_{.j.k.l.m}\Big)y^i\Big\} + \frac{8}{(n+1)^2}\{E_{jm}E_{kl} + E_{jl}E_{km} + E_{jk}E_{lm}\}y^i. \tag{25}$$

With a plain calculation, we arrive at

$$\mathbf{B^i_{jkl;m}} = -\frac{\mathbf{1}}{\mathbf{2}}\{\mathbf{R}_{.\mathbf{m.j.k}}\boldsymbol{\delta}^{\mathbf{i}}_{\mathbf{l}} + \mathbf{R}_{.\mathbf{m.j.l}}\boldsymbol{\delta}^{\mathbf{i}}_{\mathbf{k}} + \mathbf{R}_{.\mathbf{m.k.l}}\boldsymbol{\delta}^{\mathbf{i}}_{\mathbf{j}} + \mathbf{R}_{.\mathbf{m.j.k.l}}\mathbf{y^i}\} + \frac{\mathbf{8}}{(\mathbf{n}+\mathbf{1})^2}\{\mathbf{E_{jm}E_{kl}} + \mathbf{E_{jl}E_{km}} + \mathbf{E_{jk}E_{lm}}\}\mathbf{y^i}. \tag{26}$$

From the expression (26), the proof is completed.

Proposition 3.7. Let $\boldsymbol{G} = y^i\frac{\partial}{\partial x^i} - 2G^i(y)\frac{\partial}{\partial y^i}$ be a locally projectively $R-flat$ spray of isotropic curvature on an open domain $U \subset R^n$, where $G^i = Py^i$ and $P = P(x,y)$. Then, we have $\boldsymbol{H} = 0$.

Proof: We have a formula (13) for the mean Berwald curvature coefficients. Hence, we can write

$$E_{jk;m}y^m = \frac{n+1}{6}\{\tau_{m.j.k}y^m\}. \tag{27}$$

Since $\tau_m = -\frac{1}{2}R_{.m}$, the expression (27) becomes

$$E_{jk;m}y^m = -\frac{n+1}{12}\{R_{.m.j.k}y^m\}. \tag{28}$$

From the definition of $\boldsymbol{H}-$curvature and the expression (28) for the locally projectively $R-flat$ spray of isotropic curvature $\boldsymbol{G}$, we get $\boldsymbol{H} = 0$.

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